Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals (Section C)
TRENT UNIVERSITY, Fall 2021

Solutions to Quiz #3
Wednesday, 6 October.

Do all three of the following problems. Simplify your answers as much as you reasonably can.

1. Find the derivative of $f(x) = \ln(\sec(x) + \tan(x))$. *[1.5]*

SOLUTION. This is a job for the Chain Rule, plus knowledge of the derivatives of ln, sec, and tan.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\ln(\sec(x)+\tan(x)) = \frac{1}{\sec(x)+\tan(x)} \cdot \frac{d}{dx}(\sec(x)+\tan(x)) \\
&= \frac{1}{\sec(x)+\tan(x)} \cdot \left(\frac{d}{dx}\sec(x) + \frac{d}{dx}\tan(x)\right) \\
&= \frac{1}{\sec(x)+\tan(x)} \cdot \left(\sec(x)\tan(x) + \sec^2(x)\right) \\
&= \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)} = \frac{\sec(x)(\sec(x)+\tan(x))}{\sec(x)+\tan(x)} \\
&= \sec(x) \quad \blacksquare
\end{aligned}$$

2. Find the derivative of $g(x) = \dfrac{x^2 - 2x + 1}{x^2 + 2x - 3}$. *[1.5]*

SOLUTION. We will use the Quotient Rule as our main tool, but not until we have simplified the given function.

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\left(\frac{x^2-2x+1}{x^2+2x-3}\right) = \frac{d}{dx}\left(\frac{(x-1)^2}{(x-1)(x+3)}\right) = \frac{d}{dx}\left(\frac{x-1}{x+3}\right) \\
&= \frac{\left[\frac{d}{dx}(x-1)\right](x+3) - (x-1)\left[\frac{d}{dx}(x+3)\right]}{(x+3)^2} = \frac{1\cdot(x+3)-(x-1)\cdot 1}{(x+3)^2} \\
&= \frac{4}{(x+3)^2} \quad \blacksquare
\end{aligned}$$

3. Find $\dfrac{dy}{dx}$ as best you can if $e^y = \dfrac{x+y}{e^x}$. *[2]*

SOLUTION. Solving for $\dfrac{dy}{dx}$ is a little easier if we first rearrange the equation that relates x and y:

$$e^y = \frac{x+y}{e^x} \implies x + y = e^x e^y = e^{x+y}$$

Note that since $e^x > 0$ for all x, we are not accidentally multiplying by 0 on both sides of the original equation, which could make the new equation a little less useful.

We now apply the technique of implicit differentiation and take the derivative of both sides of the equation, with a little help from the Chain Rule and knowing that the derivative of the natural exponential function is itself.

$$x + y = e^{x+y} \implies \frac{d}{dx}(x+y) = \frac{d}{dx}e^{x+y} \implies 1 + \frac{dy}{dx} = e^{x+y} \cdot \frac{d}{dx}(x+y)$$
$$\implies 1 + \frac{dy}{dx} = e^{x+y} \cdot \left(1 + \frac{dy}{dx}\right)$$

At this point there are two ways to go, both of which work.

First, for the last equation to be true, we must have that $e^{x+y} = 1$ or that $1 + \frac{dy}{dx} = 0$. In the former case, we must have $x + y = 0$, so $y = -x$, and thus $\frac{dy}{dx} = -1$; in the latter case we must have $\frac{dy}{dx} = -1$ right away.

Second, we can continue from the last equation and solve for $\frac{dy}{dx}$ directly:

$$1 + \frac{dy}{dx} = e^{x+y} \cdot \left(1 + \frac{dy}{dx}\right) \implies 1 + \frac{dy}{dx} = e^{x+y} + e^{x+y} \cdot \frac{dy}{dx}$$
$$\implies \left(1 - e^{x+y}\right) \cdot \frac{dy}{dx} = e^{x+y} - 1$$
$$\implies \frac{dy}{dx} = \frac{e^{x+y} - 1}{1 - e^{x+y}} = -1$$

We'll studiously ignore the fact that this method had us divide by zero. (We know from the first way above that in fact $x + y = 0$ and so $e^{x+y} = 1$, *i.e.* $1 - e^{x+y} = 0$.)

Either way, $\frac{dy}{dx} = -1$. ■

[Total = 5]