Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2019

Solutions to Assignment #5
Approximating Definite Integrals

Due on Friday, 27 November.

You might want to skim through Chapter 7 and Section 8.6 of the textbook for fuller (if still incomplete) discussions of definite integrals and numerical approximations of same. Taking a peek at the handout *A Precise Definition of the Definite Integral* might help you appreciate both how hard it is really define definite integrals in a truly general way amd how much simplification the Right-Hand Rule below really provides.

The definite integral $\int_a^b f(x)\,dx$ represents the signed or weighted area of the region between $y = f(x)$ and the x-axis for $a \leq x \leq b$, where area above the x-axis is added and area below the x-axis is subtracted. The definite integral is usually defined in terms of limits of "Riemann sums", but the full general definition, while necessary to justify all the properties of definite integrals and to handle a pretty wide range of functions, is also pretty cumbersome to work with. For a lot of purposes, we can get by with a much simpler definition, such as the Right-Hand Rule, given below, which suffices to develop at least some of the properties of the definite integral and will, in principle, properly compute $\int_a^b f(x)\,dx$ for most commonly encountered functions. As a reminder:

RIGHT-HAND RULE. Suppose $f(x)$ is defined for all x in $[a,b]$ and is continuous at all but finitely many points of $[a,b]$. Then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\cdot\frac{b-a}{n}\right)\right] = \lim_{n\to\infty}\left[\frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\cdot\frac{b-a}{n}\right)\right]$$

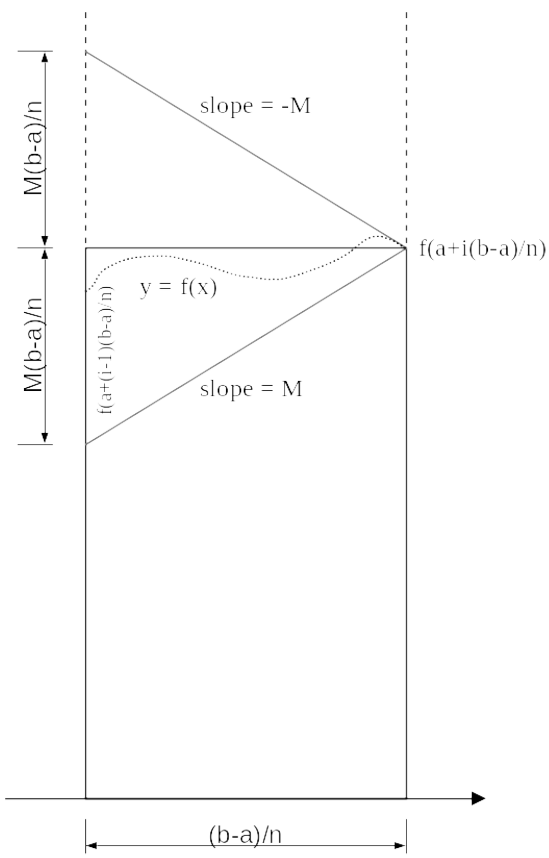
The idea here is that we divide up the interval $[a,b]$ into n subintervals of equal width $\frac{b-a}{n}$, so the ith subinterval, going from left to right and where $1 \leq i \leq n$, will be $\left[(i-1)\cdot\frac{b-a}{n}, i\cdot\frac{b-a}{n}\right]$. Each subinterval serves as the base of a rectangle of height $f\left(a + i\cdot\frac{b-a}{n}\right)$, which must then have area $\frac{b-a}{n} f\left(a + i\cdot\frac{b-a}{n}\right)$. The sum of the areas of these rectangles, the *nth Right-Hand Rule sum for* $\int_a^b f(x)\,dx$, approximates the area computed by $\int_a^b f(x)\,dx$. (It's called the Right-Hand Rule because it uses the right-hand endpoint of each subinterval to evaluate $f(x)$ at to determine the height of the rectangle which has that subinterval as a base.) As we increase n and so shrink the width of the rectangles we get better and better approximations to the definite integral. The object of this assignment is to work out how quickly the approximations get better as long as the drivative of $f(x)$ is well-behaved.

1. Suppose $|f'(x)| \leq M$ for all $x \in [a,b]$, where $M \geq 0$ is a constant. Show that

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \right| \leq \frac{M(b-a)^2}{n} \quad [5]$$

Hint: Show that the error contributed by the ith rectangle in the Right-Hand Rule sum is at most $\frac{M(b-a)^2}{n^2}$. To see how that might work, draw a picture of what is going on at the top of this rectangle. The discussion of the more sophisticated Trapezoid and Simpson's Rules in §8.6 of our textbook is a useful model here.

SOLUTION. Suppose $|f'(x)| \leq M$, *i.e.* $-M \leq f'(x) \leq M$, for all $x \in [a,b]$ for some constant $M \geq 0$ and consider a generic ith rectangle in the sum for the Right-Hand Rule with $[a,b]$ divided up into n subintervals. This rectangle has as its base the subinterval $\left[a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}\right]$, which has width $\frac{b-a}{n}$, and has its height given by $f\left(a + i\frac{b-a}{n}\right)$.



Observe that because the derivative of $f(x)$ is bounded, with $-M \leq f'(x) \leq M$, it follows that $f(x)$ can change by at most $M\frac{b-a}{n}$ over an interval of width $\frac{b-a}{n}$. It follows that the value of $f(x)$ at the left-hand endpoint of the ith subinterval, $f\left(a + (i-1)\frac{b-a}{n}\right)$, must be between the value at the right-hand endpoint, give or take M, *i.e.* $f\left(a + i\frac{b-a}{n}\right) - M \leq f\left(a + (i-1)\frac{b-a}{n}\right) \leq f\left(a + i\frac{b-a}{n}\right) + M$, as in the picture above. Note also that it follows that entire graph of $y = f(x)$ over this subinterval must be contained in the triangle with vertices $\left(a + (i-1)\frac{b-a}{n}, f\left(a + i\frac{b-a}{n}\right) - M\right)$, $\left(a + (i-1)\frac{b-a}{n}, f\left(a + i\frac{b-a}{n}\right) + M\right)$, and $\left(a + i\frac{b-a}{n}, f\left(a + i\frac{b-a}{n}\right)\right)$.

Since the area of the region between the graph of $y = f(x)$ on the ith subinterval and the top of the ith rectangle, *i.e.* the error contributed by the ith rectangle, is contained in the triangle, the area of the triangle $\frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 2M\frac{b-a}{n} \cdot \frac{b-a}{n} = \frac{M(b-a)^2}{n^2}$ is an upper bound for the error. (In fact, most of the time it would be serious overkill ...) As there are n rectangles in the Right-Hand Rule sum, it follows that the total error is bounded by n times the upper bound of the error for each individual rectangle:

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \right| \leq n \cdot \frac{M(b-a)^2}{n^2} = \frac{M(b-a)^2}{n} \qquad \square$$

2. Using the formula given in **1**, how large would n have to be to guarantee that the nth Right-Hand Rule sum for $\displaystyle\int_{-2}^{2} \left(4 - x^2\right)\,dx$ is within 0.01 of the correct value of the definite integral. *[2]*

Solution. To use the upper bound for the total error given in **1** we need to know a, b, and M. In this case $a = -2$ and $b = 2$, so $b - a = 2 - (-2) = 4$. Since $|f'(x)| = |-2x| = 2|x|$, it follows that $|f'(x)| = 2|x| \leq 2 \cdot 2 = 4$ for all $x \in [-2, 2]$ (which is to say, for all x with $|x| \leq 2$). Thus we can use $M = 4$.

If we can get the upper bound for the total error, $\frac{M(b-a)^2}{n}$, to be ≤ 0.01, it will follow that the total error (*i.e.* the difference between the nth Right-Hand Rule sum and $\int_{-2}^{2} \left(4 - x^2\right)\,dx$), is less than 0.01 too. Since

$$\frac{M(b-a)^2}{n} = \frac{4 \cdot 4^2}{n} = \frac{64}{n} \leq 0.01 \iff n \geq \frac{64}{0.01} = 6400\,,$$

it follows that having n be at least 6400 will guarantee that the corresponding Right-Hand Rule sum is within 0.01 of the actual value of the definite integral $\displaystyle\int_{-2}^{2} \left(4 - x^2\right)\,dx$. $\square$

3. Use `Maple` to compute both $\displaystyle\int_{-2}^{2} \left(4 - x^2\right)\,dx$ and the nth Right-Hand Rule sum for this definite integral for the n you worked out in your answeer to **2**. Is the difference between them indeed at most $\dfrac{M(b-a)^2}{n^2}$, using the M and n from your solution to **2**? *[3]*

Hint: If using `Maple`'s worksheet mode, you'll want to look up the `int` and `sum` operators.

Solution. Please see the appended worksheet.

We can compute $\int_{-2}^{2} (4 - x^2)\, dx$ by typing `int(4-x^2,x=-2..2)` into `Maple`'s worksheet mode, giving the answer $\frac{32}{3}$.

To compute the Right-Hand Rule sum for this definite integral with $b - a = 4$, $M = 4$ and $n = 6400$, as obtained in the solution to **2** above, namely

$$\frac{2-(-2)}{6400} \sum_{i=1}^{6400} \left(4 - \left(a + i \cdot \frac{2-(-2)}{6400} \right)^2 \right) = \frac{4}{6400} \sum_{i=1}^{6400} \left(4 - \left(a + i \cdot \frac{4}{6400} \right)^2 \right),$$

we type `4/6400*sum((4-(-2+i*4/6400)^2,i=1..6400)` into our `Maple` worksheet. This gives the answer $\frac{13653333}{1280000}$.

To find the difference between the two values, we use `Maple`'s handy `evalf` command, which evaluates expressions, putting the answers in decimal or scientific notation. Typing in `evalf(32/3-13653333/1280000)` gives the answer $2.604166667\ 10^{-7}$, *i.e.* approximately 0.00000026. Note that this difference is a *lot* smaller than 0.01. ∎

> $int(4 - x^2, x = -2..2)$

$$\frac{32}{3} \qquad (1)$$

> $\frac{4}{6400} \cdot sum\left(4 - \left(-2 + \frac{i \cdot 4}{6400}\right)^2, i = 1..6400\right)$

$$\frac{13653333}{1280000} \qquad (2)$$

> $evalf\left(\frac{32}{3}\right)$

$$10.66666667 \qquad (3)$$

> $evalf\left(\frac{13653333}{1280000}\right)$

$$10.66666641 \qquad (4)$$

> $evalf\left(\frac{32}{3} - \frac{13653333}{1280000}\right)$

$$2.604166667\ 10^{-7} \qquad (5)$$

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