Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Fall 2020

Solutions to Quiz #1
Tuesday, 22 September.

Do *both* of the following problems:

1. Use the ε–δ definition of limits* to verify that $\lim_{x \to 0} \frac{x}{1+\sin^2(x)} = 0$. *[2.5]*

SOLUTION. I prefer the standard version of the ε–δ definition of limits, so that's what I'll use. Thus, to verify that $\lim_{x \to 0} \frac{x}{1+\sin^2(x)} = 0$ we must check that for any $\varepsilon > 0$, there is a $\delta > 0$, such that for any x satisfying $|x - 0| < \delta$ we have $\left| \frac{x}{1+\sin^2(x)} - 0 \right| < \varepsilon$.

Suppose, then, that we are given an $\varepsilon > 0$. As usual, we try to reverse-engineer a corresponding δ from the desired conclusion:

$$\begin{aligned} \left| \frac{x}{1+\sin^2(x)} - 0 \right| < \varepsilon &\iff \frac{|x|}{1+\sin^2(x)} < \varepsilon \qquad \text{(Since } 1+\sin^2(x) \geq 1+0 = 1 > 0.\text{)} \\ &\iff |x| < \varepsilon\left(1+\sin^2(x)\right) \end{aligned}$$

Obviously, we can't use $\delta = 1 + \sin^2(x)$, since δ may not depend on x. However, all we need is to find a $\delta > 0$ so that if $|x - 0| = |x| < \delta$, then $|x| < \varepsilon\left(1+\sin^2(x)\right)$. Since $\sin^2(x) = [\sin(x)]^2 \geq 0$, we have (as already noted above) that $1+\sin^2(x) \geq 1+0 = 1$, so $\delta = \varepsilon$ will do:

$$|x - 0| < \delta = \varepsilon \iff |x| < \varepsilon \leq \varepsilon\left(1+\sin^2(x)\right) \implies \left| \frac{x}{1+\sin^2(x)} - 0 \right| = \frac{|x|}{1+\sin^2(x)} < \varepsilon$$

It follows that $\lim_{x \to 0} \frac{x}{1+\sin^2(x)} = 0$ by the (standard) ε–δ definition of limits. $\square$

NOTE: The argument using the game version of the ε–δ definition of limits is essentially at the same at the key steps. Verifying the limit means finding a winning strategy for player B, the one who plays δ in the second move of the limit game. Finding a winning strategy boils down to responding to player A's $\varepsilon > 0$ with a $\delta > 0$ such that for any x with $|x - 0| < \delta$ we have $\left| \frac{x}{1+\sin^2(x)} - 0 \right| < \varepsilon$, which leads to exactly the reverse-engineering problem the standard definition does.

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* Whichever of the standard or game version happens to be your preference.

2. Compute† $\displaystyle\lim_{x\to 1} \frac{1-\sqrt{x^2+2x+2}}{(x+1)^2}$. *[2.5]*

SOLUTION. Well, all the operations used to construct the function we're taking the limit of are continuous wherever they are defined. As long as we don't end up dividing by 0 or something like that, we can compute the function by evaluating the function at the point we are taking the limit. (This is by the Continuity Rule for evaluating limits; rule 7 in the list given in the *Limits III* lecture.) Let's try it:

$$\lim_{x\to 1} \frac{1-\sqrt{x^2+2x+2}}{(x+1)^2} = \frac{1-\sqrt{1^2+2\cdot 1+2}}{(1+1)^2} = \frac{1-\sqrt{5}}{2^2}\,\frac{1-\sqrt{5}}{4}$$

No problems! Not much fuss or muss either ... :-) ■

NOTE. If you're feeling cheated by getting a problem that was too easy, you can always try $\displaystyle\lim_{x\to -1} \frac{1-\sqrt{x^2+2x+2}}{(x+1)^2}$, which is a bit more challenging.

† Using algebra and the practical rules for computing limits. You do *not* have to verify you are correct using the ε–δ definition of limits. (Unless you're a mathochist ... :-)