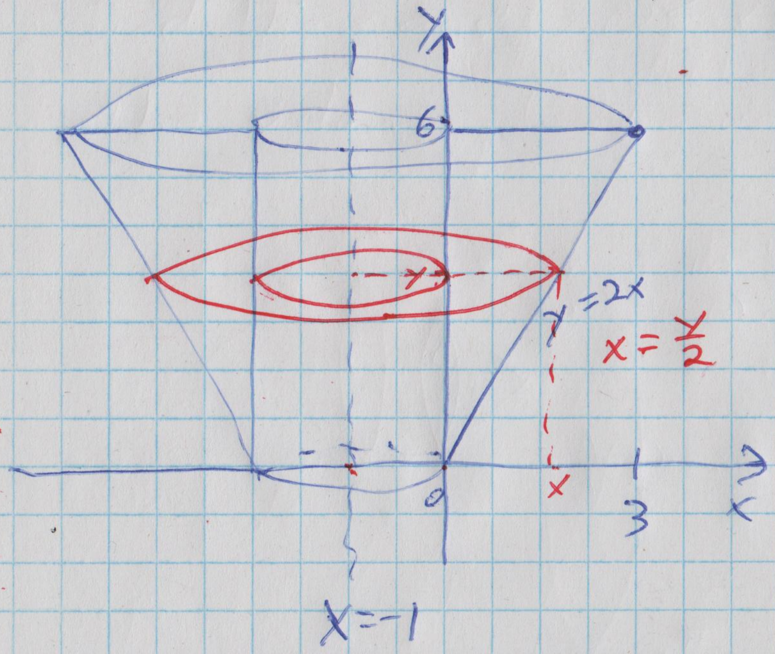
Volumes II - Solids of Revolution

2020-12-06 ①



Revolve the region between $y=2x$ & the y-axis, for $0 \le x \le 3$, about the line $x=-1$ & find the volume of the resulting solid.

$$V = \int_0^6 (\pi R^2 - \pi r^2)\,dy$$

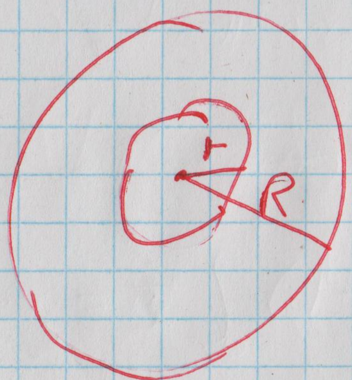
$$= \pi \int_0^6 (R^2 - r^2)\,dy = \pi \int_0^6 \left(\left(\frac{y}{2}+1\right)^2 - 1^2 \right) dy$$

$$= \pi \int_0^6 \left(\frac{y^2}{4} + \cancel{2}\frac{y}{\cancel{2}} + \cancel{1} - \cancel{1} \right) dy$$

$$= \pi \int_0^6 \left(\frac{y^2}{4} + y \right) dy$$

$$= \pi \left(\frac{y^3}{4 \cdot 3} + \frac{y^2}{2} \right) \Bigg|_0^6 = \pi \left(\frac{6^{\cancel{3}\,2}}{\cancel{12}\,2\cdot\cancel{6}} + \frac{6^2}{2} \right) - \pi \cancel{\left(\frac{0^3}{12} + \frac{0^2}{2} \right)}$$

$$= \pi \left(\frac{36}{2} + \frac{36}{2} \right) = 36\pi.$$



"Disk/Washer" method for computing volumes of solids of revolution.

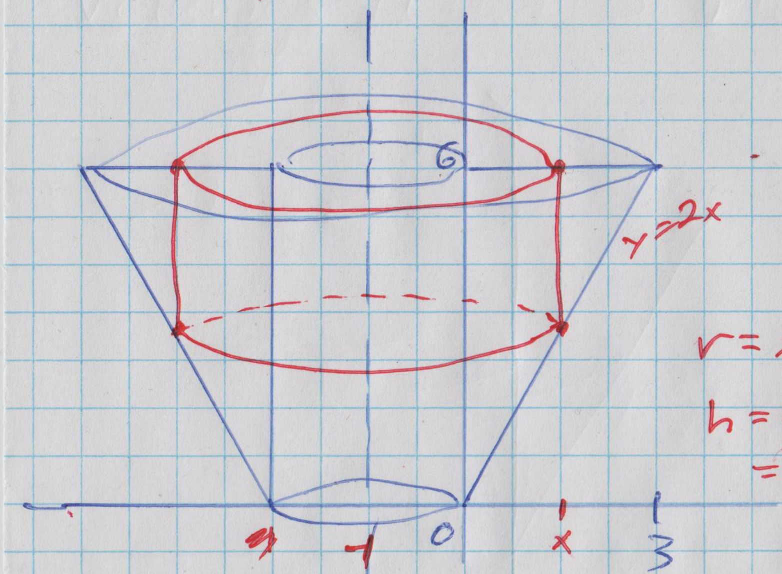
$r = 1$

$R = x - (-1)$

$= x + 1$

$= \frac{y}{2} + 1$

Alternate method: cylindrical shells for the cross-sections



$r = x-(-1) = x+1$

$h = 6 - y$
$= 6 - 2x$

$$V = \int_0^3 A(x)\,dx = \int_0^3 2\pi r h\,dx$$

$$= \int_0^3 2\pi(x+1)(6-2x)\,dx$$

$$= 2\pi\int_0^3 (6x - \cancel{1}2x^2 + 6 - 2x)\,dx$$

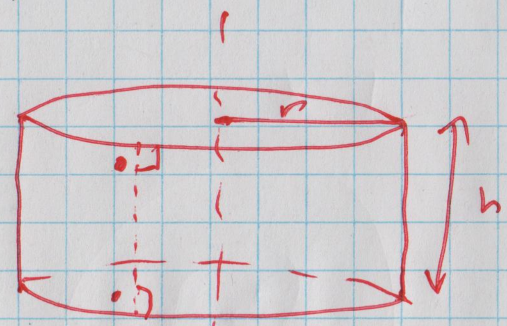
$$= 2\pi\int_0^3 (-2x^2 + 4x + 6)\,dx$$

$$= 2\pi\left(-\frac{2}{3}x^3 + 4\frac{x^2}{2} + 6x\right)\Big|_0^3$$

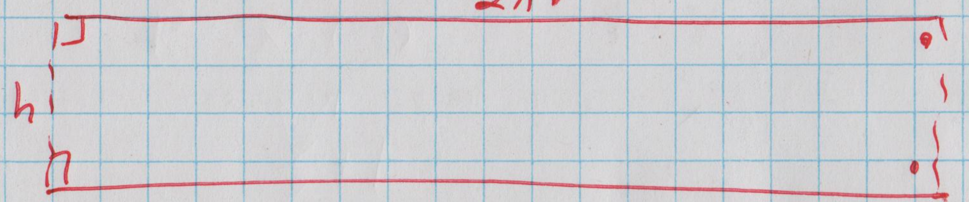
$$= 2\pi\left(-\frac{2}{3}3^3 + 2\cdot 3^2 + 6\cdot 3\right) - 2\pi(\cdots)$$

$$= 2\pi(\cancel{-2\cdot 9 + 2\cdot 9} + 18) - 0$$

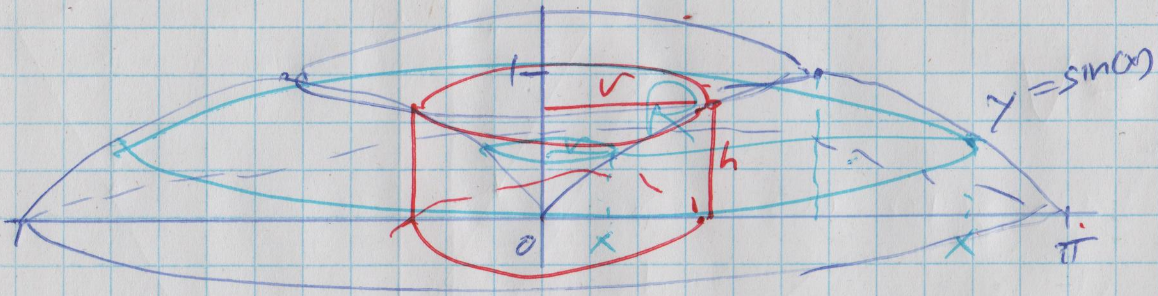
$$= 36\pi \checkmark$$



Area of a cylinder of radius r & height h (without caps) is $2\pi r h$.



Example: (where cylindrical shells are much easier)



Revolve the region between $y = \sin(x)$ & the x-axis about the y-axis

Washers: $R = x = r$?

$r = \arcsin(y)$

$R = \cancel{\arcsin(y)} + \pi - \arcsin(y)$

$\pi \int (R^2 - r^2)\,dx$

How do we integrate $\arcsin(y)$ & so on...

Hard!

Shells: $r = x$

$h = \sin(x) - 0$

$$V = \int_0^{\pi} 2\pi r h\,dx = \int_0^{\pi} 2\pi x \sin(x)\,dx$$

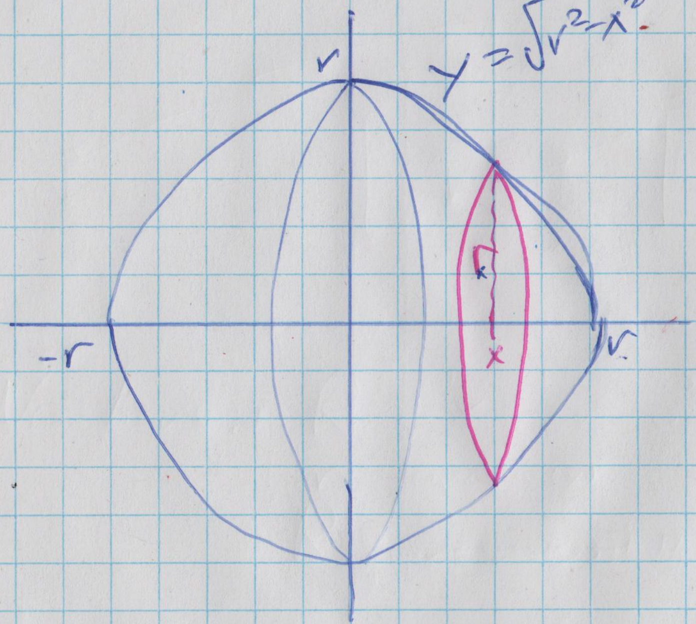
$$= 2\pi \left[-x\cos(x)\Big|_0^{\pi} + \int_0^{\pi} 1\cdot(+\cos(x))\,dx \right] \quad \begin{array}{ll} u = x & v' = \sin(x) \\ u' = 1 & v = -\cos(x) \end{array}$$

$$= 2\pi \left[-\pi\cdot\cos(\pi) - 0\cdot\cos(0) + \sin(x)\Big|_0^{\pi} \right]$$

$$= 2\pi \left[-\pi(-1) - \cancel{0\cdot 1} + \cancel{\overset{0}{\sin(\pi)}} - \cancel{\sin(0)} \right]$$

$$= 2\pi^2$$

One more example: Find the volume of a sphere of radius r.



Get the sphere:

Revolve the half-disk between $y=\sqrt{r^2-x^2}$

for $-r \le x \le r$, about the x-axis

Disk/Washer method: $r_x = y = \sqrt{r^2-x^2}$

Use x since the disks are perpendicular to the x-axis

$$V=\int_{-r}^{r} \pi r_x^2\,dx = \int_{-r}^{r} \pi\left(\sqrt{r^2-x^2}\right)^2 dx$$

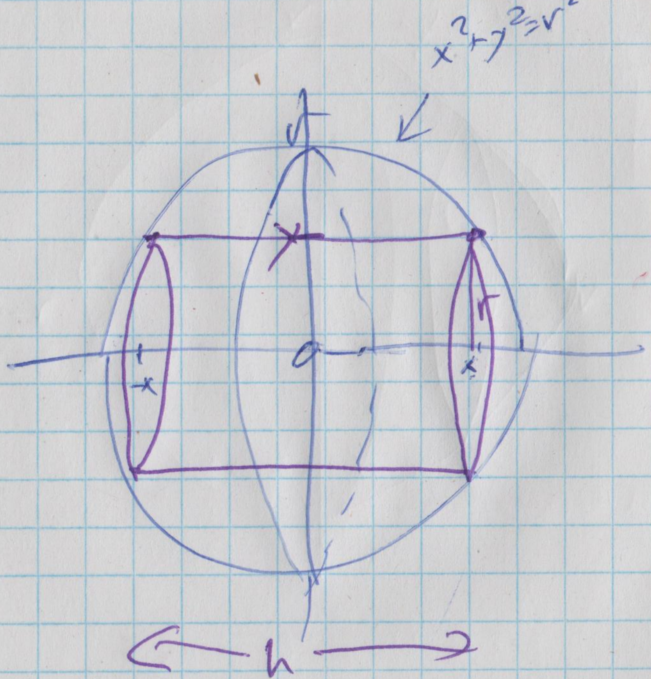
$$= \int_{-r}^{r} \pi(r^2-x^2)\,dx = \pi\left(r^2x - \frac{x^3}{3}\right)\Big|_{-r}^{r}$$

$$= \pi\left(r^2\cdot r - \frac{r^3}{3}\right) - \pi\left(r^2(-r) - \frac{(-r)^3}{3}\right)$$

$$= \pi\left(\frac{2}{3}r^3\right) + \pi\left(\cancel{-}\frac{2}{3}r^3\right)$$

$$= \frac{4}{3}\pi r^3$$

With shells: We need to use y as the variable, since the y-axis is perpendicular to the shells



$h = x-(-x) = 2x$

$r_y = y$

$h = 2x = 2\sqrt{r^2-y^2}$

$A(y) = 2\pi r h$

$= 2\pi y \cdot 2\sqrt{r^2-y^2}$

$$V = \int_0^r A(y)\,dy = \int_0^r 4\pi y\sqrt{r^2-y^2}\,dy$$

$u = r^2-y^2$

$du = -2y\,dy$

$(-1)du = 2y\,dy$

y	u
0	r^2
r	0

$$= 2\pi \int_{r^2}^{0} \sqrt{u}\cdot(-1)\,du$$

$$= 2\pi \int_0^{r^2} u^{1/2}\,du = 2\pi \cdot \frac{u^{3/2}}{3/2}\Big|_0^{r^2}$$

$$= \frac{4}{3}\pi (r^2)^{3/2} - \cancel{\frac{4}{3}\pi\, 0^{3/2}}$$

$$= \frac{4}{3}\pi r^3 \quad \checkmark$$