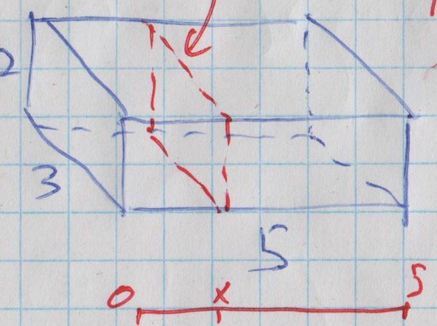
Volumes

Example:

cross-section at x has area $A(x) = 2\cdot 3 = 6$

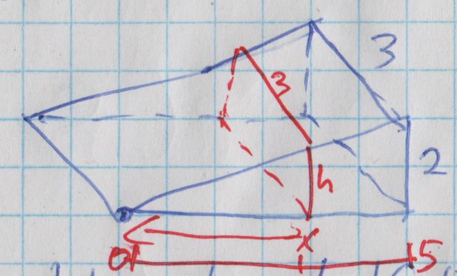


We compute volumes of solids by integrating the areas of cross-sections. ("Areas sweep out volumes.")

$$V_{box} = l\cdot w\cdot h = 5\cdot 3\cdot 2 = 30 \text{ units}^2$$

$$V_{box} = \int_0^5 A(x)\,dx = \int_0^5 6\,dx = 6x\Big|_0^5 = 6\cdot 5 - 6\cdot 0 = 30$$

Half the box as a wedge:



$$\frac{h}{x} = \frac{2}{5} \Rightarrow h = \frac{2}{5}x$$

$$A(x) = 3\cdot\frac{2}{5}x = \frac{6}{5}x$$

...this has half the volume, ie $\frac{1}{2}30 = 15$.

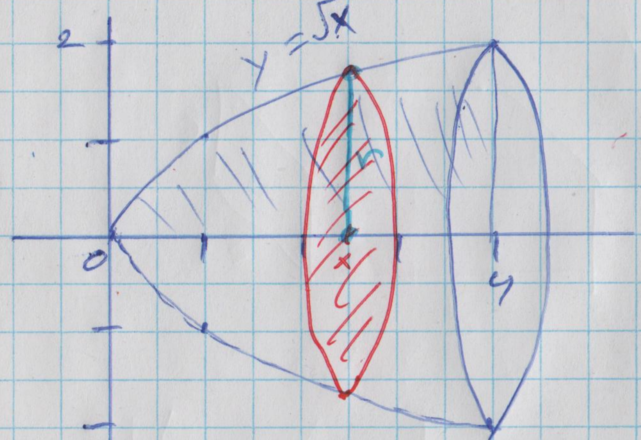
Notice the cross-sections are perpendicular to x...

$$V_{wedge} = \int_0^5 A(x)\,dx = \int_0^5 \frac{6}{5}x\,dx = \frac{6}{5}\cdot\frac{x^2}{2}\Big|_0^5$$

$$= \frac{3}{5}x^2\Big|_0^5 = \frac{3}{5}\cdot 5^2 - \frac{3}{5}\cdot 0^2$$

$$= 3\cdot 5 - 0 = 15$$

Problem: What is the volume of the solid obtained by revolving the region below $y = \sqrt{x}$ and above $y = 0$, for $0 \le x \le 4$, about the x-axis?



"paraboloid"

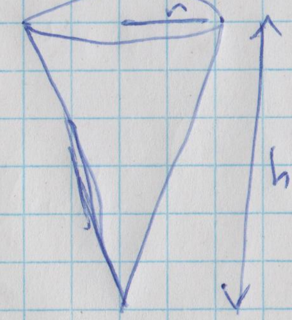
The cross-sections perpendicular to the x-axis are disks.

The disk at x has area πr^2, where $r = \sqrt{x}$, thus $A(x) = \pi(\sqrt{x})^2 = \pi x$.

$$V_{paraboloid} = \int_0^4 A(x)\,dx = \int_0^4 \pi x\,dx = \frac{\pi x^2}{2}\Big|_0^4$$

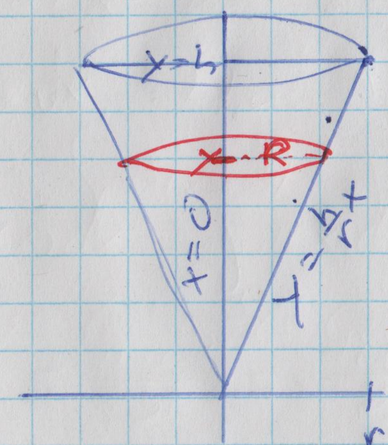
$$= \frac{\pi \cdot 4^2}{2} - \cancel{\frac{\pi \cdot 0^2}{2}} = \frac{16\pi}{2} = 8\pi$$

Problem: Suppose we have a cone of height h and radius at the top of r. What is the volume of the cone in terms of r & h?



$V = ?$

We can get this cone by revolving the triangle ~~[illegible]~~ whose borders are pieces of the lines $x=0$ (y-axis), $y=h$, & $y=\frac{h}{r}\cdot x$.



$0 \le y \le h$

Let "big" R be the radius of the cross-section at x

$y = \frac{h}{r}x$

$\Rightarrow R = x = \frac{ry}{h}$

So $A(y) = \pi R^2 = \pi\left(\frac{ry}{h}\right)^2 = \frac{\pi r^2}{h^2}\cdot y^2$

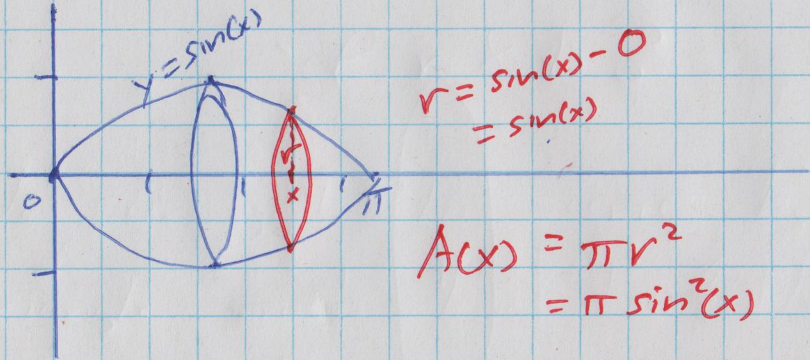
$$V_{cone} = \int_0^h A(y)\,dy = \int_0^h \pi R^2\,dy$$

$$= \int_0^h \frac{\pi r^2}{h^2}\cdot y^2\,dy = \frac{\pi r^2}{h^2}\cdot\frac{y^3}{3}\Big|_0^h$$

$$= \frac{\pi r^2}{h^2}\cdot\frac{h^3}{3} - \cancel{\frac{\pi r^2}{h^2}\cdot\frac{0^3}{3}}$$

$$= \frac{\pi r^2 h}{3} = \frac{1}{3}\pi r^2 h \quad \checkmark$$

Q: Suppose that we revolve the region between $y = \sin(x)$ and the x-axis, $0 \le x \le \pi$, about the x-axis. What is the volume of the resulting solid?



$r = \sin(x) - 0$
$= \sin(x)$

$A(x) = \pi r^2$
$= \pi \sin^2(x)$

$$V = \int_0^\pi A(x)\,dx = \int_0^\pi \pi r^2\,dx$$

$$= \boxed{\int_0^\pi \pi \sin^2(x)\,dx}$$

Use parts!

$u = \sin(x)$

$v' = \sin(x)$

$u' = \cos(x)$

$v = -\cos(x)$

$$= -\pi \sin(x)\cos(x)\Big|_0^\pi + \pi \int_0^\pi \cos(x)(+\cos(x))\,dx$$

$$= \cancel{-\pi(-\overset{0}{\sin(\pi)}\cos(\pi))} \cancel{-\pi(-\overset{0}{\sin(0)}\cos(0))} + \pi\int_0^\pi \cos^2(x)\,dx$$

$$= \pi \int_0^\pi (1 - \sin^2(x))\,dx = \pi \int_0^\pi 1\,dx - \pi\int_0^\pi \sin^2(x)\,dx = \pi x\Big|_0^\pi - \pi \int_0^\pi \sin^2(x)\,dx$$

$$= \pi\cdot\pi - \pi\cdot 0 - \boxed{\pi\int_0^\pi \sin^2(x)\,dx} \quad \therefore \quad 2\boxed{\pi\int_0^\pi \sin^2(x)\,dx} = \pi^2$$

$$\therefore \quad V = \boxed{\int_0^\pi \pi \sin^2(x)\,dx} = \frac{\pi^2}{2}$$