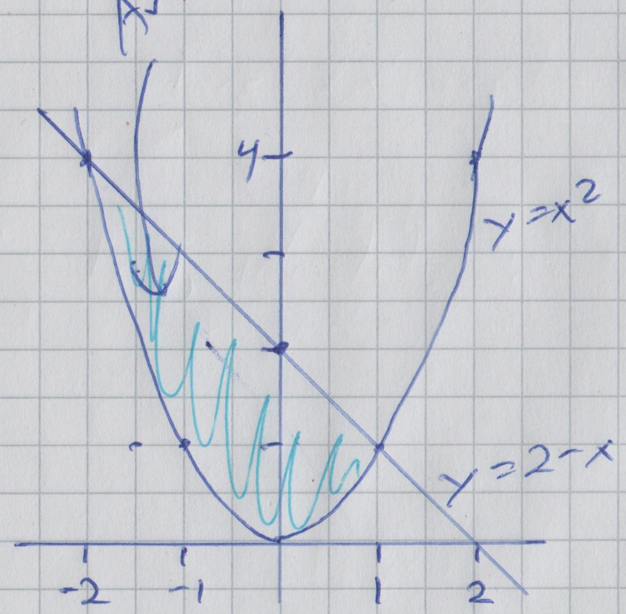
2020-11-27

Areas Between Curves - an application of integration

Problem: Find the area of the parabolic segment below the line $y=2-x$ and above the parabola $y=x^2$.

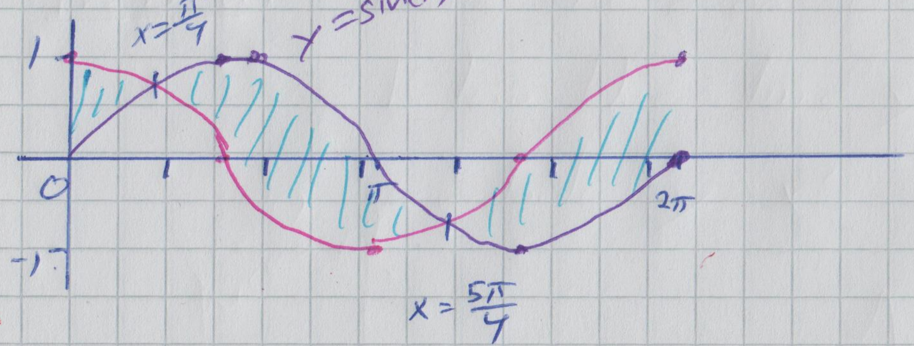


$$\text{Area} = \int_{-2}^{1} (\text{upper function} - \text{lower function})\,dx$$
$$= \int_{-2}^{1} \left((2-x) - x^2\right) dx$$
$$= \int_{-2}^{1} (2-x-x^2)\,dx$$
$$= \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-2}^{1}$$
$$= \left(2\cdot 1 - \frac{1^2}{2} - \frac{1^3}{3}\right) - \left(2\cdot(-2) - \frac{(-2)^2}{2} - \frac{(-2)^3}{3}\right)$$
$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 - \left(-\frac{8}{3}\right)\right)$$
$$= \frac{7}{6} - \left(\frac{-10}{3}\right) = \frac{7}{6} + \frac{20}{6} = \frac{27}{6} = \boxed{\frac{9}{2}}$$
$$= 4.5$$

- obviously, the curves intersect at $x=-2$ ($\& \ y=4$) and at $x=1$ ($\& \ y=1$)

[Could solve $x^2-(2-x)=0$ for x.]

Problem: Find the area ($\overset{\text{not the weighted area}}{\downarrow}$) between the curves $y = \sin(x)$ and $y = \cos(x)$ for $0 \le x \le 2\pi$.



These curves start out with $\cos(x)$ above $\sin(x)$ at $x=0$, cross at $x = \frac{\pi}{4}$, so $\sin(x)$ get above $\cos(x)$, cross again at $x = \frac{5\pi}{4}$, and $\cos(x)$ get above again, until the region ends at $x = 2\pi$.

$$\int_0^{2\pi} (\text{upper} - \text{lower})\,dx$$ Breakup accordingly!

$$= \int_0^{\pi/4} (\cos(x) - \sin(x))\,dx + \int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\,dx + \int_{5\pi/4}^{2\pi} (\cos(x) - \sin(x))\,dx$$

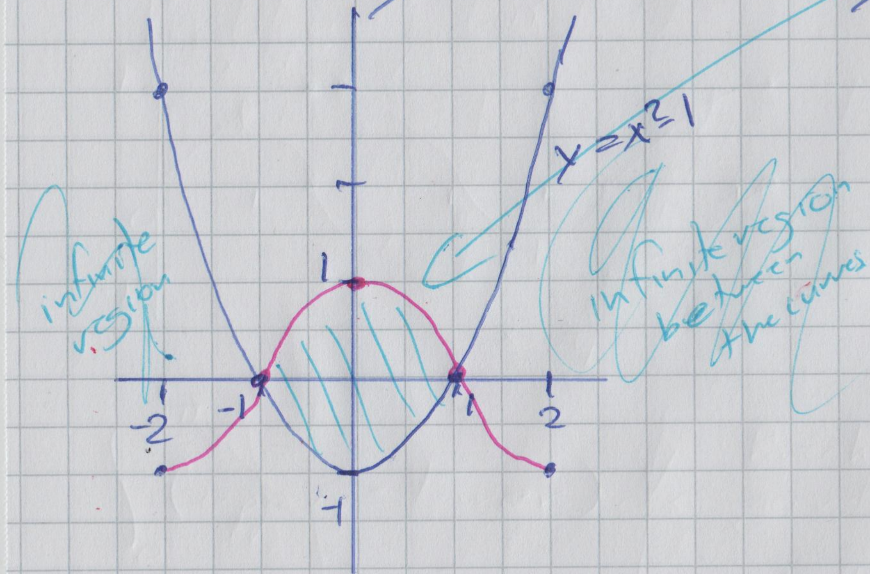
$$= \left(\sin(x) - (-\cos(x))\right)\Big|_0^{\pi/4} + \left(-\cos(x) - \sin(x)\right)\Big|_{\pi/4}^{5\pi/4} + \left(\sin(x) - (-\cos(x))\right)\Big|_{5\pi/4}^{2\pi}$$

$$= \left(\sin(x) + \cos(x)\right)\Big|_0^{\pi/4} - \left(\cos(x) + \sin(x)\right)\Big|_{\pi/4}^{5\pi/4} + \left(\sin(x) + \cos(x)\right)\Big|_{5\pi/4}^{2\pi}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)\right] - \left[\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right] + \left[(0+1) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$$

$$= \left[\frac{2}{\sqrt{2}} \cancel{-1}\right] - \left[-\frac{4}{\sqrt{2}}\right] + \left[\cancel{1} + \frac{2}{\sqrt{2}}\right] = \frac{8}{\sqrt{2}} = \boxed{4\sqrt{2}}$$

Problem: Find the area of the finite region between $y = x^2 - 1$ and $y = \cos\left(\frac{\pi x}{2}\right)$.



Notice that the region runs from $x = -1$ to $x = 1$, and $y = \cos\left(\frac{\pi}{2}x\right)$ is above $y = x^2 - 1$ on this region.

$$\int_{-1}^{1} (\text{upper} - \text{lower})\,dx = \int_{-1}^{1}\left(\cos\left(\frac{\pi x}{2}\right) - (x^2 - 1)\right)dx$$

$$= \int_{-1}^{1}\cos\left(\frac{\pi}{2}x\right)dx - \int_{-1}^{1}(x^2 - 1)\,dx$$

$u = \frac{\pi}{2}x$

$\Rightarrow du = \frac{\pi}{2}\cdot dx$ & $\begin{array}{c|c} x & u \\ \hline -1 & -\pi/2 \\ 1 & \pi/2 \end{array}$

$\Rightarrow dx = \frac{2}{\pi}\,du$

$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\cos(u)\,du - \left(\frac{x^3}{3} - x\right)\Big|_{-1}^{1}$$

$$= \frac{2}{\pi}\sin(x)\Big|_{-\pi/2}^{\pi/2} - \left[\left(\frac{1}{3} - 1\right) - \left(\frac{-1}{3} - (-1)\right)\right]$$

$$= \left[\frac{2}{\pi}\cdot 1 - \frac{2}{\pi}(-1)\right] - \left[\left(-\frac{2}{3}\right) - \left(\frac{2}{3}\right)\right]$$

$$= \left[2\cdot\frac{2}{\pi}\right] - \left[-\frac{4}{3}\right] = \boxed{\frac{4}{\pi} + \frac{4}{3}} = 4\left(\frac{1}{\pi} + \frac{1}{3}\right)$$