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Examples of Substitution II

Reminder: The Substitution Rule is

$$\int f(g(x)) \cdot g'(x)\,dx \qquad u = g(x), \quad du = g'(x)\,dx \qquad \text{(Indefinite Integral form)}$$

$$= \int f(u)\,du$$

& eventually substitute to put the antideravite in terms of x

$$\int_a^b f(g(x)) \cdot g'(x)\,dx \qquad u = g(x), \quad du = g'(x) \qquad \text{(Definite Integral form)}$$

$$= \int_{x=a}^{x=b} f(u)\,du$$

& eventually substitute back before using the limits

or

$$= \int_{g(a)}^{g(b)} f(u)\,du$$

& don't have to substitute back before using the <u>new</u> limits.

Warm-up: $\int_0^1 \frac{1}{(2x+3)^2}\,dx$

$u = 2x+3$

$du = (2x+3)'\,dx = 2\,dx$

$\frac{1}{2}du = dx$

x	u
0	3
1	5

$$= \int_3^5 \frac{1}{u^2}\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_3^5 u^{-2}\,du \overset{\text{Power Rule}}{=} \frac{1}{2}\cdot \frac{u^{-2+1}}{-2+1}\Big|_3^5 = \frac{-1}{2}u^{-1}\Big|_3^5$$

$$= -\frac{1}{2u}\Big|_3^5 = \left(-\frac{1}{2\cdot 5}\right) - \left(-\frac{1}{2\cdot 3}\right) = -\frac{1}{10} + \frac{1}{6}$$

$$= -\frac{3}{30} + \frac{5}{30} = \frac{2}{30} = \boxed{\frac{1}{15}}$$

Example: (This has one has multiple possibilities for substitution.)

① $\int \frac{\ln(z^2)}{z}\,dz$ Try $u = z^2$, so $du = 2z\,dz$ & $z\,dz = \frac{1}{2}du$

$\Rightarrow \frac{1}{2z}du = dz$

$$= \int \frac{\ln(u)}{z}\cdot\frac{1}{2z}\,du = \int \frac{\ln(u)}{2z^2}\,dz = \frac{1}{2}\int \frac{\ln(u)}{u}\,du$$

$w = \ln(u)$

$dw = \frac{1}{u}du$

must put all z in terms of u

$$= \frac{1}{2}\int w\,dw = \frac{1}{2}\cdot\frac{w^2}{2} + C = \frac{1}{4}[\ln(u)]^2 + C$$

$$= \frac{1}{4}[\ln(z^2)]^2 + C$$

2. The first was a mess - but we made it work!

This time we'll try a more sensible approach...

$$\int \frac{\ln(z^2)}{z}\,dz \qquad s = \ln(z^2) \qquad \frac{ds}{dz} = \frac{1}{z^2}\cdot\frac{d}{dz}(z^2) = \frac{1}{z^2}\cdot 2z = \frac{2}{z}$$

so $ds = \frac{2}{z}dz$, and so $\frac{1}{z}dz = \frac{1}{2}ds$

$$= \int s\cdot\frac{1}{2}\,ds = \frac{1}{2}\cdot\frac{s^2}{2} + C = \frac{s^2}{4} + C = \frac{1}{4}\left[\ln(z^2)\right]^2 + C$$

3. Do a little prep to simplify the integrand first:

$$\int \frac{\ln(z^2)}{z}\,dz = \int \frac{2\ln(z)}{z}\,dz \qquad t = \ln(z),\ \text{so } dt = \frac{1}{z}dz$$

$$= 2\int t\,dt = \cancel{2}\cdot\frac{t^2}{\cancel{2}} + C = \boxed{\left[\ln(z)\right]^2 + C}$$

Looks different from $\frac{1}{4}\left[\ln(z^2)\right]^2 + C$, but

$$= \frac{1}{4}\left[2\ln(z)\right]^2 + C$$

$$= \cancel{\tfrac{1}{4}}\cdot\cancel{4}\cdot\left[\ln(z)\right]^2 + C$$

so it's really the same.

A more challenging example:

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$$ What do we substitute for?

$$= \int \frac{(e^x)^2}{\sqrt{e^x+1}}\,dx$$ Try to simplify by substituting $u = e^x$, so $du = e^x dx$

$$= \int \frac{e^x \cdot e^x \overset{du}{dx}}{\sqrt{e^x+1}} = \int \frac{u}{\sqrt{u+1}}\,du$$ Substitute $w = u+1$, so $dw = du$ & $u = w-1$

$$= \int \frac{w-1}{\sqrt{w}}\,dw = \int\left(\frac{w}{\sqrt{w}} - \frac{1}{\sqrt{w}}\right)dw = \int\left(\sqrt{w} - \frac{1}{\sqrt{w}}\right)dw$$

$$= \int\left(w^{1/2} - w^{-1/2}\right)dw = \frac{w^{3/2}}{3/2} - \frac{w^{1/2}}{1/2} + C = \frac{2}{3}w^{3/2} - \frac{1}{2}w^{1/2} + C$$

$$= \frac{2}{3}(u+1)^{3/2} - \frac{1}{2}(u+1)^{1/2} + C = \frac{2}{3}(e^x+1)^{3/2} - \frac{1}{2}(e^x+1)^{1/2} + C.$$

$$= \frac{2}{3}\left(\sqrt{e^x+1}\right)^3 - \frac{1}{2}\sqrt{e^x+1} + C$$

An alternative substitution for this problem would be to whole hog and substitute $y = \sqrt{e^x+1}$

Easier to solve for dx by solving for x first: $y^2 = e^x + 1$

$\Rightarrow e^x = y^2 - 1$

$\Rightarrow \cancel{x = \ln\left(\sqrt{y^2-1}\right)}$

$$\frac{dx}{dy} = \frac{1}{\sqrt{y^2-1}} \cdot \frac{d}{dy}\sqrt{y^2-1}$$

$$= \frac{1}{\sqrt{y^2-1}} \cdot \frac{d}{dy}(y^2-1)^{1/2}$$

$$= \frac{1}{\sqrt{y^2-1}} \cdot \frac{1}{2}(y^2-1)^{-1/2} \cdot \frac{d}{dy}(y^2-1)$$

$$= \frac{1}{\sqrt{y^2-1}} \cdot \frac{1}{2\sqrt{y^2-1}} \cdot 2y$$

$$= \frac{y}{y^2-1}$$

$$\therefore dx = \frac{y}{y^2-1}\, dy$$

$$x = \ln(y^2-1)$$

$$dx = \frac{1}{y^2-1} \cdot \frac{d}{dy}(y^2-1) \cdot dy = \frac{1}{y^2-1} \cdot 2y \cdot dy = \frac{2y}{y^2-1}\, dy$$

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\, dx$$

$$= \int \frac{(y^2-1)^{\cancel{2}}}{\cancel{y}} \cdot \frac{2\cancel{y}}{\cancel{y^2-1}}\, dy$$

$$= \int 2(y^2-1)\, dy = 2\frac{y^3}{3} - 2y + C$$

$$= \frac{2}{3}\left(\sqrt{e^x+1}\right)^3 - 2\sqrt{e^x+1} + C$$

$\uparrow$

Q: Why did we get a 2 here instead of $\frac{1}{2}$ as we did before? Exercise: Find out.