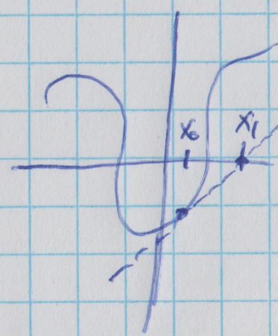
Two More Applications

Newton's Method

– an algorithm for finding approximations to the zeros of a function,

i.e. find an x s.t. $f(x) = 0$.



Suppose we start with some x value x_0.

Consider the point $(x_0, f(x_0))$ and the slope $m = f'(x_0)$.

The tangent line passing through that point has the slope m, so it has an equation of the form

$$\frac{y - f(x_0)}{x - x_0} = m = f'(x_0) \Rightarrow y - f(x_0) = m(x - x_0)$$

$$\Rightarrow y = mx - mx_0 + f(x_0)$$

$$= f'(x_0) \cdot x - f'(x_0) \cdot x_0 + f(x_0)$$

$$\Rightarrow 0 = f'(x_0) \cdot x_1 - f'(x_0) \cdot x_0 + f(x_0)$$

$$\Rightarrow x_1 = \frac{f'(x_0) \cdot x_0 - f(x_0)}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

"Lather, rinse, repeat."

Similarly, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$, etc.

In most cases, if the function has a zero,

$\left[x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \right]$ then the sequence x_n homes in on one.

ie $\lim_{n\to\infty} x_n = x$ for some x such that $f(x)=0$.

$\left[\text{Since } \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x, \text{ so } x = x + \frac{f(x)}{f'(x)} \Rightarrow f(x) = 0. \right]$

eg $f(x) = x^2 - 5$ (So a solution to $f(x)=0$, is $\pm\sqrt{5}$.)

$x_0 = 2$ $\quad f'(x) = 2x$

$$x_1 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{2^2-5}{2\cdot 2} = 2 - \frac{-1}{4} = 2 + \frac{1}{4} = \frac{9}{4} = 2.25$$

$$x_2 = \frac{9}{4} - \frac{f(\frac{9}{4})}{f'(\frac{9}{4})} = \frac{9}{4} - \frac{(\frac{9}{4})^2 - 5}{2\cdot\frac{9}{4}} = \frac{9}{4} - \frac{\frac{81}{16} - 5}{\frac{9}{2}} = \frac{9}{4} - \left(\frac{81}{16} - 5\right)\cdot\frac{2}{9}$$

$$= \frac{9}{4} - \left(\frac{9}{8} - \frac{10}{9}\right) = \frac{162}{72} - \left(\frac{81}{72} - \frac{80}{72}\right) = \frac{162}{72} - \frac{1}{72} = \frac{161}{72}$$

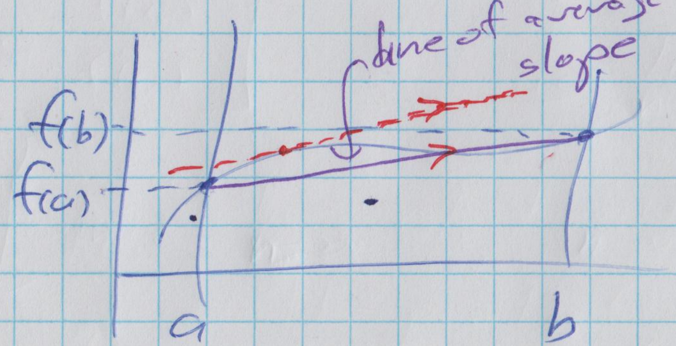
$$= 2.2361111\ldots$$

This already very close to $\sqrt{5} = 2.236067\ldots$.

This is the ancestor of the methods used by Maple et al. to approximate roots.

Mean Values (Average values)

The average slope of a continuous function $f(x)$ on $[a,b]$ is



$$m = \frac{f(b)-f(a)}{b-a}.$$

This means that the derivative has the property that for some $a<c<b$, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Mean Value Theorem

If $f(x)$ is continuous on $[a,b]$, and differentiable on (a,b), then there is a $c \in (a,b)$ such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

eg This allows us to estimate ~~that~~ how much a function can change on an interval if we have a bound on the derivative.

We'll prove the Mean Value Theorem.

<u>Lemma:</u> (Rolle's Theorem) Suppose $f(x)$ is cts. on $[a,b]$, and diff'ble on (a,b), and $f(a) = f(b)$. Then there is a $c \in (a,b)$ such that $f'(c) = 0$.

<u>proof:</u> Since $f(x)$ is continuous on $[a,b]$, it has a maximum and a minimum on this interval.

This max and/or min must occur either at the endpoint(s) or at a critical point.

If the max (or min) occurs at a critical point, $c \in (a,b)$ then $f'(c) = 0$, so we've done.

Otherwise, both max & min must occur at the endpoints, ie $f(a) = f(b) = 0$ is both the max & min.

This means that $f(x) = 0$ for all $x \in [a,b]$, but then $f'(c) = 0$ for all $c \in (a,b)$. //

proof (of the Mean Value Theorem):

Suppose $f(x)$ is cts. on $[a,b]$ & diff'ble on (a,b).

Let $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a} \cdot (x-a)$.

Then $g(x)$ is also cts. on $[a,b]$ and diff'ble. on (a,b).

$g(a) = \cancel{f(a) - f(a)} - \frac{\cancel{f(b)-f(a)}}{b-a} \cdot \cancel{(a-a)} = 0$

and $g(b) = \cancel{f(b)} - f(a) - \frac{\cancel{f(b)} - f(a)}{\cancel{b-a}} \cdot \cancel{(b-a)} = 0$.

So by Rolle's Theorem, there is a $c \in (a,b)$ s.t. $g'(c) = 0$.

But $g'(x) = f'(x) - 0 - \frac{f(b)-f(a)}{b-a} \cdot (1-0)$

so $g'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a}$

ie $f'(c) = \frac{f(b)-f(a)}{b-a}$, as required. //

Next time: Integration.