Related Rates II - Three dimensions home

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1° A spherical balloon is inflated at a rate of 2L/min. How is the diameter of the balloon changing at the instant that the volume of the balloon is 4L?

<u>Facts:</u> $V_{sphere} = \frac{4}{3}\pi r^3$ where r is the radius of the sphere

D_{sphere} = diameter = $2r$ — " —

$$\frac{dV}{dt} = 2L/min = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \underbrace{\frac{4}{3}\pi \cdot 3r^2}_{\frac{dV}{dr}} \cdot \frac{dr}{dt}$$

$$\boxed{\frac{dD}{dt} = \frac{d}{dt}(2r) = 2\frac{dr}{dt}}$$

$$\Rightarrow \frac{dr}{dt} = \frac{2}{\frac{4}{\cancel{3}}\pi \cdot \cancel{3}r^2} = \frac{1}{2\pi r^2}$$

But we don't know r ...

At the instant in question: $4 = V = \frac{4}{3}\pi r^3 \Rightarrow 1 = \frac{\pi}{3}r^3$

$$\Rightarrow r^3 = \frac{3}{\pi} \Rightarrow r = \sqrt[3]{\frac{3}{\pi}} = \left(\frac{3}{\pi}\right)^{1/3}$$

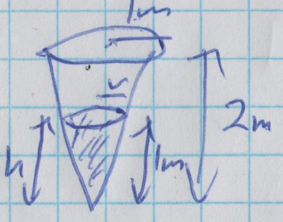
$$\therefore \left.\frac{dr}{dt}\right|_{V=4} = \frac{1}{2\pi\left[\left(\frac{3}{\pi}\right)^{1/3}\right]^2} = \frac{1}{2\pi\frac{3^{2/3}}{\pi^{2/3}}} = \frac{1}{2\pi^{1/3}3^{2/3}} \text{ dm/min}$$

$(1L = 1\,dm^3)$

∴ The diameter of the balloon is changing at a rate of $\frac{dD}{dt} = 2\frac{dr}{dt} = 2 \cdot \frac{1}{2\pi^{1/3}3^{2/3}} = \frac{1}{\pi^{1/3}3^{2/3}}$ dm/min

$= \frac{10}{\pi^{1/3}3^{2/3}}$ cm/min.

2° A right circular conical tank has height 2m and a radius at the circular end of 1m. It is oriented point down and filled with water. The water is then drained out of the tank at a rate of 1 L/s. (At any given instant the water in the tank has a conical shape.) How is the depth of the water in the tank changing at the instant that the depth is 1m?



$V_{cone} = \frac{1}{3}\pi r^2 h$ where r is the radius of the circular end & h is the height of the cone.

$\frac{dV}{dt} = -1$ L/s $\quad \left.\frac{dh}{dt}\right|_{h=1} = ?$

By similarity (the cone of water in the tank has the same proportions as the tank does)

$$\frac{r}{h} = \frac{1}{2} \Rightarrow r = \frac{h}{2} \quad \therefore \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{h^2}{2^2} h$$
$$= \frac{\pi}{12} h^3$$

$$-1\,L/s = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi}{12}h^3\right) = \frac{d}{dh}\left(\frac{\pi}{12}h^3\right) \cdot \frac{dh}{dt}$$
$$= \frac{\pi}{12} \cdot 3h^2 \cdot \frac{dh}{dt} = \frac{\pi}{4}h^2 \cdot \frac{dh}{dt}$$

$$\therefore \quad \frac{dh}{dt} = \frac{-1}{\frac{\pi}{4} \cdot h^2} = \frac{-4}{\pi h^2}$$

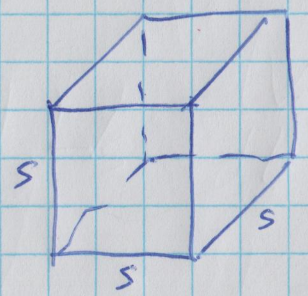
At the instant that $h = 1\,m$, $\frac{dh}{dt} = \frac{-4}{\pi \cdot 1^2} = -\frac{4}{\pi}$ ~~m/s~~.

What's wrong with this? We messed up the units - we need to be consistent - since $1\,L \neq 1\,m^3 = 1000\,L$.

Measuring h in m means we should measure volume in m^3

$-1\,L/s = -0.001\,m^3/s$ ie $\frac{dh}{dt} = \frac{-0.004}{\pi h^2}$, so $\left.\frac{dh}{dt}\right|_{h=1} = \frac{-0.004}{\pi \cdot 1^2} = \frac{-0.004}{\pi}\,m/s$

3° A Borg cubes' volume expands by $1\,m^3$ for every 100 kg of matter it ingests. If it ingests matter at a constant rate of 3000 kg/s, how quickly is its surface are changing at the instant that its sides are 10 m each?



$V = s^3 \qquad SA = 6s^2$

We know that $\frac{dV}{dt} = 3000\,kg/s \cdot \frac{1\,m^3}{100\,kg}$

$= 30\,m^3/s$.

$$\frac{dV}{dt} = \frac{dV}{ds} \cdot \frac{ds}{dt} = 3s^2 \frac{ds}{dt} \quad \therefore \quad \frac{ds}{dt} = \frac{30}{3s^2} = \frac{10}{s^2}\ m/s$$

$$\frac{dSA}{dt} = \frac{dSA}{ds} \cdot \frac{ds}{dt} = \frac{d}{ds}(6s^2) \cdot \frac{ds}{dt} = 12s \cdot \frac{10}{s^2} = \frac{120}{s}\ m^2/s$$

When $s = 10\,m$, we get $\left.\frac{dSA}{dt}\right|_{s=10} = \frac{120}{10}\ m^2/s = 12\,m^2/s$.