Curve Sketching II

2020-10-20

The "curve-sketching" (qualitative analysis) checklist:

0° Intercepts

1° Vertical Asymptotes

2° Horizontal Asymptotes

3° Increase / Decrease & Local Max/Min ("Slope")

4° Curvature & Pts. of Inflection

5° Sketch of the graph based on 0°–4°

We'll illustrate the process using
$$f(x) = \begin{cases} e^{-1/x^2} & x<0 \\ 0 & x=0 \\ xe^{-x} & x>0 \end{cases}.$$

0° Intercepts: y-int. Set $x=0$. $f(0)=0$ [so also an x-int.]

x-int. Set $y=0$ & solve for x. $f(x)=0$?

$\boxed{f(0)=0}$, Only x-int. $f(x)=e^{-1/x^2} \neq 0$ for all $x<0$, and $f(x)=xe^{-x} \neq 0$ for all $x>0$ ...

1° VA: We need to check any points where the definition of f fails entirely or gets funky.

For $f(x)$, this means checking at $x=0$.

(Note that e^{-1/x^2} is defined & cts. for all $x<0$,

and $x \cdot e^{-x}$ — " — , for all $x>0$.)

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{-1/x^2} = 0^+$$

As $x\to 0^-$,

$x^2 \to 0^+$,

$\frac{1}{x^2} \to +\infty$,

$-\frac{1}{x^2} \to -\infty$,

so $e^{-1/x^2} \to 0^+$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \underset{\substack{\downarrow \\ 0^+}}{x}\,\underset{\substack{\downarrow \\ 1}}{e^{-x}} = 0^+ \cdot e^{-0} = 0^+ \cdot 1 = 0^+$$

$\therefore$ We have no VA at $x=0$ either.

Note that $\lim_{x\to 0^-} f(x) = 0 = f(0) = \lim_{x\to 0^+} f(x)$, so

$f(x)$ is continuous at $x=0$.

2° Horizontal Asymptotes

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} e^{-1/x^2} = 1^-$$

As $x \to -\infty$,
$x^2 \to +\infty$,
$\frac{1}{x^2} \to 0^+$,
$-\frac{1}{x^2} \to 0^-$,
$e^{-1/x^2} \to 1^-$.

So we have a HA of $y = \cancel{0}\ 1$ in the negative direction, which is approached from below.

$$\lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \underset{\infty}{x}\,\underset{0}{e^{-x}} = \lim_{x \to \infty} \frac{x \to \infty}{e^x \to \infty}$$

Use l'Hopital's Rule

$$= \lim_{x \to \infty} \frac{\frac{d}{dx} x}{\frac{d}{dx} e^x} = \lim_{x \to \infty} \frac{1 \to 1}{e^x \to \infty} = 0^+$$

So we have a HA of $y = 0$ in the the positive direction, which is approached from above.

3° Slope & max/min

Look for critical points:

When $x<0$, $f(x) = e^{-1/x^2}$, so
$$f'(x) = \frac{d}{dx} e^{-1/x^2}$$
$$= e^{-1/x^2} \cdot \frac{d}{dx}\left(\frac{-1}{x^2}\right)$$
$$= e^{-1/x^2} \cdot \frac{(-1)(-2)}{x^3}$$
$$= \frac{2e^{-1/x^2}}{x^3} = 0 ?$$

only when? never!
Since $e^t > 0$ for all t.

When $x>0$, $f(x) = xe^{-x}$, so
$$f'(x) = \frac{d}{dx}(xe^{-x})$$
$$= \left(\frac{d}{dx}x\right)(e^{-x}) + (x)\left(\frac{d}{dx}e^{-x}\right)$$
$$= 1 \cdot e^{-x} + x(-e^{-x})$$
$$= (1-x)\underbrace{e^{-x}}_{>0} = 0$$

exactly when $x=1$
($1>0$ so it's applicable).

When $x=0$, $f(x)$ is patched together so we check it too
(note that it is cts. at $x=0$)

When $x<0$, $f'(x) = \dfrac{\overset{>0}{2e^{-1/x^2}}}{\underset{<0}{x^3}} < 0$,

so $f(x)$ is decreasing for all $x<0$.

When $x>0$, $f'(x) = (1-x)\underset{>0}{e^{-x}}$

>0 when $1-x>0$, i.e. $x<1$

$=0$ — " — $1-x=0$, i.e. $x=0$

<0 when $1-x<0$, i.e. $x>1$.

$\therefore$ ~~Pote~~ The critical points are $x=0$ & $x=1$.

x	$(-\infty,0)$	0	$(0,1)$	1	$(1,\infty)$
$f'(x)$	$-$	?	$+$	0	$-$
$f(x)$	$\downarrow$	cts. at $x=0$ 0 local min	$\uparrow$	$\frac{1}{e}$ local max.	$\downarrow$

$1\cdot e^{-1} = \frac{1}{e}$

$\therefore$ We have a local minimum (of 0) at $x=0$ & a local maximum (of $\frac{1}{e}$) at $x=1$.

4° Curvature & inflection points

When $x<0$, $f''(x) = \frac{d^2}{dx^2} e^{-1/x^2} = \frac{d}{dx}\left(2x^{-3}e^{-1/x^2}\right)$

$$= 2(-3)x^{-4}e^{-1/x^2} + 2x^{-3}e^{-1/x^2}(2x^{-3})$$

$$= -6x^{-4}e^{-1/x^2} + 4x^{-6}e^{-1/x^2}$$

$$= 2x^{-4}e^{-1/x^2}(-3+2x^{-2})$$

$$= \underset{\geq 0}{\frac{2}{x^4}}\, \underset{\geq 0}{e^{-1/x^2}} \left(\frac{2}{x^2} - 3\right) \quad \begin{matrix} >0 \text{ if } x > -\sqrt{\frac{2}{3}} \\ <0 \text{ if } x < -\sqrt{\frac{2}{3}} \end{matrix}$$

$= 0$ when $\frac{2}{x^2} - 3 = 0$

ie $3x^2 - 2 = 0$

ie $x^2 = \frac{2}{3}$ ie $x = \pm\sqrt{\frac{2}{3}}$

Since $x<0$, $x = -\sqrt{\frac{2}{3}}$.

So we have an inflection point at $x = -\sqrt{\frac{2}{3}}$.

When $x>0$, $f''(x) = \frac{d^2}{dx^2}(xe^{-x}) = \frac{d}{dx}(1-x)e^{-x} = (-1)\cdot e^{-x} + (1-x)(-e^{-x})$

$$= -e^{-x} - e^{-x} + xe^{-x} = (x-2)\underset{}{e^{-x}}^{>0}.$$

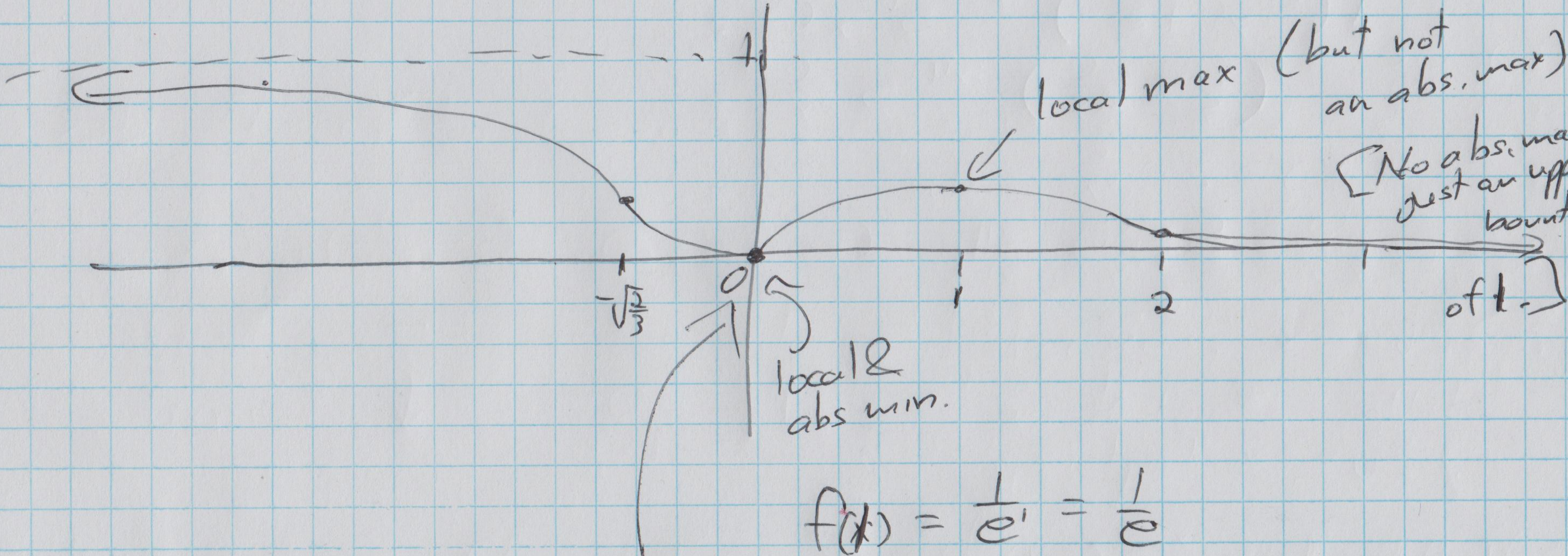
$= 0$ when $x = 2$ (>0)

< 0 when $x<2$

> 0 when $x>0$

x	$(-\infty, -\sqrt{\frac{2}{3}})$	$-\sqrt{\frac{2}{3}}$	$(-\sqrt{\frac{2}{3}}, 0)$	0	$(0,2)$	2	$(2,\infty)$
$f''(x)$	$-$	0	$+$	?	$-$	0	$+$
$f(x)$	$\cap$	infl. pt.	$\cup$	infl. pt.	$\cap$	infl. pt.	$\cup$

5°



$$f(1) = \frac{1}{e^1} = \frac{1}{e}$$

We probably don't have a well-defined derivative at $x=0$ because the graph looks like it has a corner.