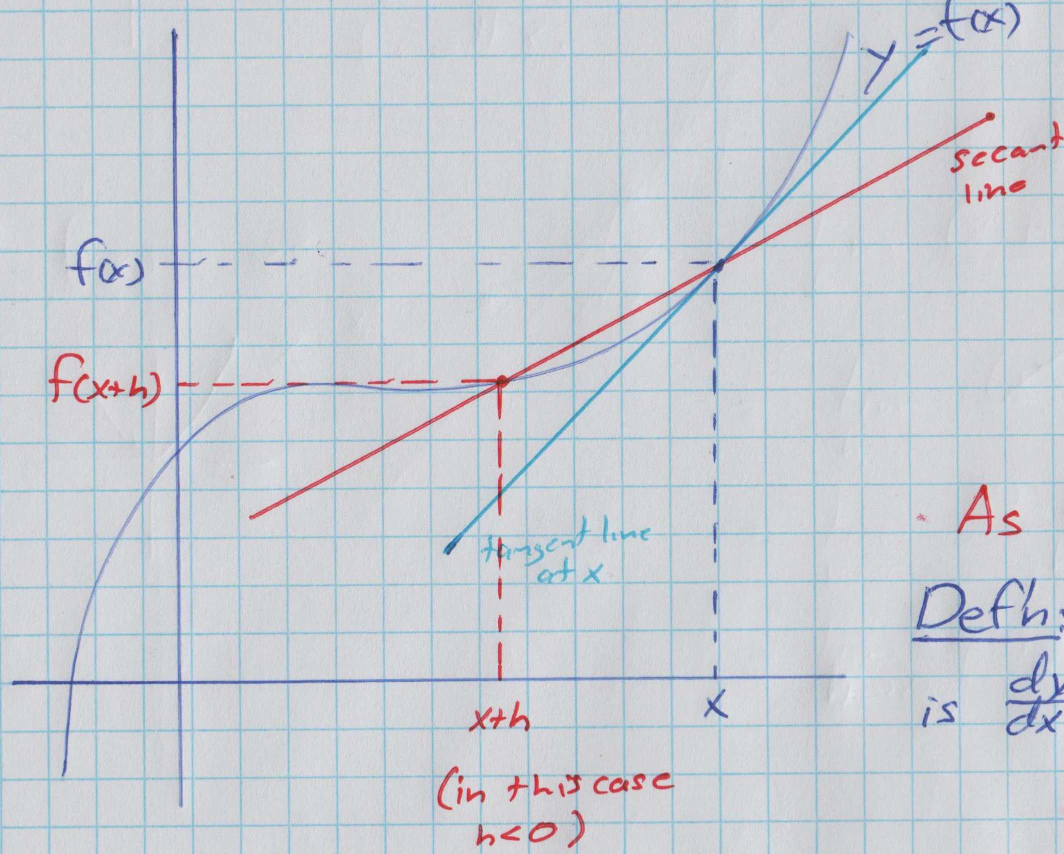
Derivatives - The limit definition thereof

Informally, the derivative of a function tells you the instantaneous rate of change of the function (at some point). In particular, the instantaneous slope of $y = f(x)$ [i.e. the slope of the tangent line at] some point is given by the derivative.



slope of the secant line is

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x+h)}{x - (x+h)} = \frac{f(x) - f(x+h)}{-h} = \frac{f(x+h) - f(x)}{h}$$

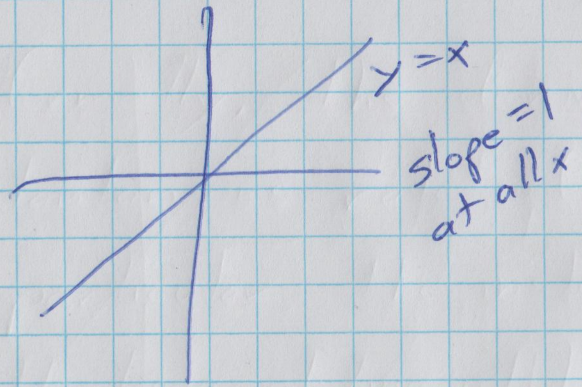
As $h \to 0$ $\frac{\Delta y}{\Delta x} \to$ slope of the tangent

<u>Defn:</u> The <u>derivative of $f(x)$ at x</u> is

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

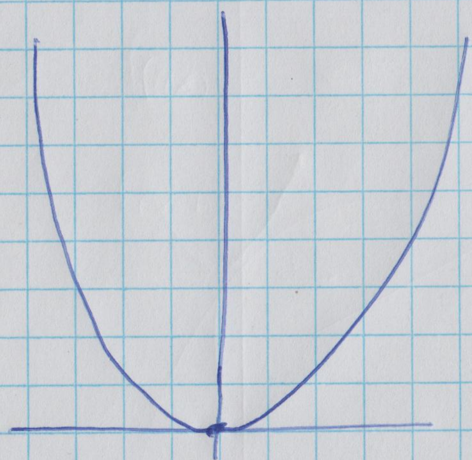
$$= \dot{y}.$$

1° Suppose $f(x) = x$.



$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}(x) = \frac{dx}{dx}$$
$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(\cancel{x}+h) - \cancel{x}}{h} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}} = \lim_{h\to 0} 1 = 1$$

2° Suppose $f(x) = x^2$.



$$f'(x) = \lim_{h\to 0} \frac{\overbrace{(x+h)^2}^{f(x+h)} - \overbrace{x^2}^{f(x)}}{h} = \lim_{h\to 0} \frac{(\cancel{x^2}+2hx+h^2) - \cancel{x^2}}{h}$$
$$= \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$
$$= \lim_{h\to 0} (2x+h) = 2x+0 = 2x \checkmark$$

$$f'(x) = \frac{d}{dx} x^2 = 2x$$

3° Suppose $f(x) = x^n$, where $n > 1$.

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$k! = k\cdot(k-1)\cdot(k-2)\cdots 3\cdot 2\cdot 1$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^n} + nhx^{n-1} + \binom{n}{2}h^2x^{n-2} + \cdots + h^n - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left(nx^{n-1} + h\binom{n}{2}x^{n-2} + \cdots + h^{n-1}\right)}{\cancel{h}}$$

$$= \lim_{h\to 0} \left(nx^{n-1} + \underbrace{h\binom{n}{2}x^{n-2}}_{\to 0} + \underbrace{\cdots}_{\to 0} + \underbrace{h^{n-1}}_{\to 0}\right)$$

$$= nx^{n-1} + 0 = nx^{n-1}$$

Thus we have the Power Rule for derivatives:

$$\frac{d}{dx}x^n = nx^{n-1}$$

(This actually works for every real power of x.
i.e. $\frac{d}{dx}x^a = ax^{a-1}$.)
$a \in \mathbb{R}$

4°

(4)

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$\boxed{\begin{array}{l} \text{eg} \\ \sin(2x) = \sin(x+x) \\ = 2\sin(x)\cos(x) \end{array}}$$

$$= \lim_{h\to 0}\frac{(\sin(x)\cos(h)+\sin(h)\cos(x)) - \sin(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{\sin(x)\cos(h)-\sin(x)}{h} + \frac{\sin(h)\cos(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\sin(x)\cdot\frac{\cos(h)-1}{h} + \cos(x)\cdot\frac{\sin(h)}{h}\right]$$

$$= \left[\lim_{h\to 0}\sin(x)\cdot\frac{\cos(h)-1}{h}\right] + \left[\lim_{h\to 0}\cos(x)\cdot\frac{\sin(h)}{h}\right]$$

$$= \sin(x)\cdot\underbrace{\boxed{\lim_{h\to 0}\frac{\cos(h)-1}{h}}}_{=0 \text{ [on faith!]}} + \cos(x)\cdot\underbrace{\boxed{\lim_{h\to 0}\frac{\sin(h)}{h}}}_{=1 \text{ [Shown in Limits V]}}$$

$$= \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x)$$

5°

$$\frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = e^x \cdot \boxed{\lim_{h \to 0} \frac{e^h - 1}{h}} = 1$$

(Limits V)

$$= e^x \cdot 1 = e^x$$

6° (<u>Sum Rule</u> for derivatives)

$$\frac{d}{dx}(f(x) + g(x)) = \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= (f+g)'(x) = \lim_{h \to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}$$

$$= \left[\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right] + \left[\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\right]$$

$$= f'(x) + g'(x) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

7° (Constant (multiple) Rule for derivatives)

c a constant

$$\frac{d}{dx}\left(cf(x)\right) = (cf)'(x) = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h}$$

$$= \lim_{h\to 0} \frac{c\left(f(x+h) - f(x)\right)}{h}$$

$$= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= cf'(x) = c\frac{d}{dx} f(x)$$

[6° & 7° say that derivatives are "linear".]

Next time: More rules
& more functions!