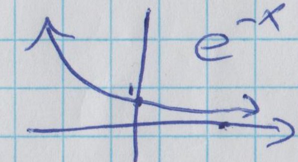
2020-09-15

1

Limits IV – Computing limits

Working through various examples using what we know (and sometimes discovering we'll need to know more).

1° $\lim_{x\to 0} x e^{-x} = 0\cdot e^{-0} = 0\cdot e^{0} = 0\cdot 1 = 0$



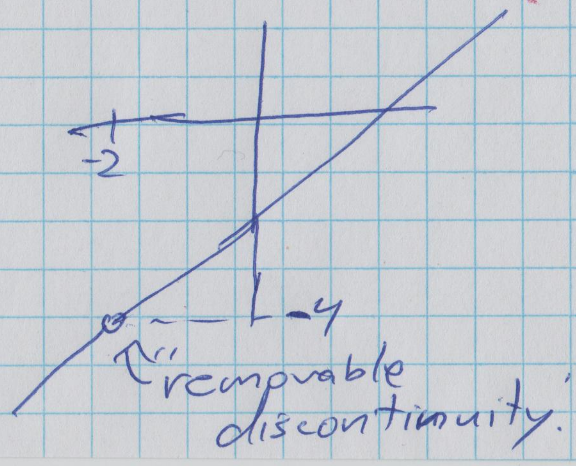
2° $\lim_{x\to -2} \left(x^2+\sqrt{x^2+x}+3x\right) = (-2)^2+\sqrt{(-2)^2+(-2)}+3(-2)$

$= 4+\sqrt{4-2}+(-6)$

$= 4+\sqrt{2}+(-6) = -2+\sqrt{2}$

3° $\lim_{x\to -2} \frac{x^2-4}{x+2} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}} = \lim_{x\to -2} (x-2) = -2-2 = -4$

$\cancel{\frac{(-2)^2+4}{-2+2} \stackrel{?}{=} \frac{4+4}{0}}$



②

$4^\circ \quad \lim_{x\to 3} \frac{x^3-27}{x^2-9} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{(x-3)}(x+3)}$

$$\begin{array}{r} x^2+3x+9 \\ x-3 \overline{)\, x^3+0x^2+0x-27} \\ -(x^3-3x^2) \\ \hline 3x^2+0x \\ -(3x^2-9x) \\ \hline 9x-27 \\ -(9x-27) \\ \hline 0 \end{array}$$

$= \lim_{x\to 3} \frac{x^2+3x+9}{x+3}$

$= \frac{3^2+3\cdot 3+9}{3+3} = \frac{9+9+9}{6} = \frac{27}{6} = \frac{9}{2}$

$= 4.5$

~~$\sqrt{a+b} = \sqrt{a}+\sqrt{b}$~~

a^2-b^2
$=(a-b)(a+b)$

$5^\circ \quad \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$

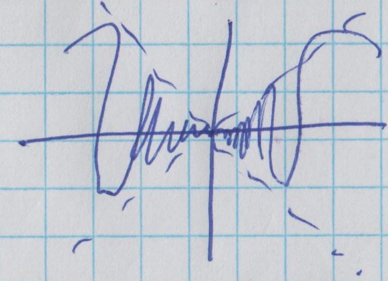
$= \lim_{t\to 0} \frac{(\sqrt{t^2+9})^2-3^2}{t^2(\sqrt{t^2+9}+3)}$

$= \lim_{t\to 0} \frac{(t^2+\cancel{9})-\cancel{9}}{t^2(\sqrt{t^2+9}+3)} = \lim_{t\to 0} \frac{\cancel{t^2}}{\cancel{t^2}(\sqrt{t^2+9}+3)}$

$= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} = \frac{1}{\sqrt{0^2+9}+3} = \frac{1}{3+3} = \boxed{\frac{1}{6}}$

6° $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

$y = \sin\left(\frac{1}{x}\right)$



$y = x\sin\left(\frac{1}{x}\right)$

Since $-1 \leq \sin(t) \leq 1$,

$-x \leq x \sin\left(\frac{1}{x}\right) \leq x$.

As $x \to 0$, this means that

$$-x \leq x\sin\left(\tfrac{1}{x}\right) \leq x$$
$$\searrow \quad \downarrow \quad \swarrow$$
$$0$$

Thus $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$ by the Squeeze Theorem since $-x \to 0$ and $x \to 0$ as $x \to 0$.

7° $\lim_{x \to \infty} \frac{1}{x} = 0$ because $\frac{1 \to 1}{x \to \infty} \to 0$

$$\lim_{x \to -\infty} \frac{1}{x} = \lim_{t \to \infty} \frac{1}{-t} = -\lim_{t \to \infty} \frac{\overset{\to 1}{1}}{\underset{\to \infty}{t}} \overset{\to 0}{} = -0 = 0$$

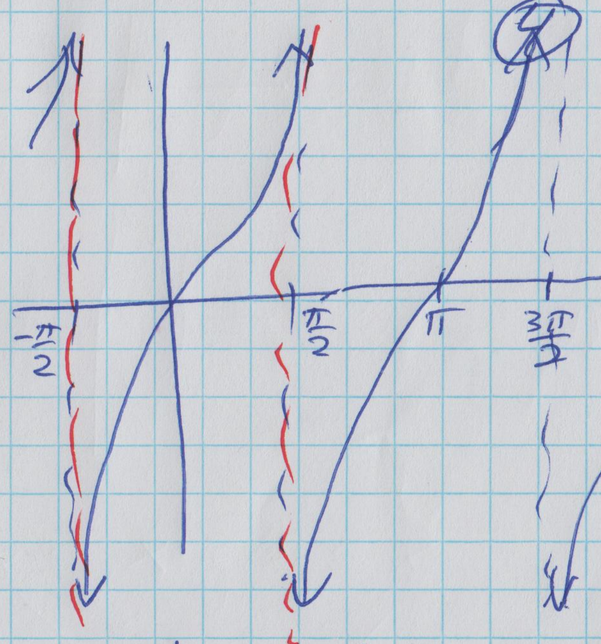
$x = -t$

8°

$$\lim_{x\to\infty} \arctan(x) = \cancel{0}\ \frac{\pi}{2}$$

$$\arctan(x) = \tan^{-1}(x)$$

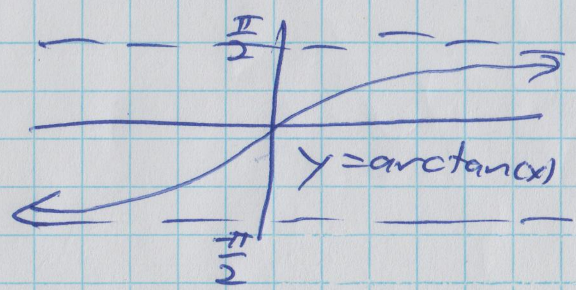
because



&

$$\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$$

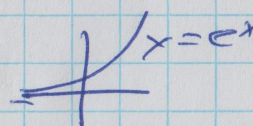
arctan inverts the 'branch' of tan that passes through 0



9°

$$\lim_{t\to\infty} e^{-t} = \lim_{t\to\infty} \frac{1}{e^t} \begin{matrix}\to 1\\ \to\infty\end{matrix}$$

$$= 0$$



$$\lim_{t\to-\infty} e^{-t} = \lim_{u\to\infty} e^{-(-u)} = \lim_{u\to\infty} e^{u}$$

$$(t=-u) \qquad = \infty$$

10° $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} \begin{matrix}\to\infty\\ \to\infty\end{matrix} = ?$

Two ways: 1° e^x (and exponentials to a base > 1) grows much faster than any polynomial. Thus $\frac{x}{e^x}$ should go to 0 as $x\to\infty$.

exponentials dominate polynomials dominate logarithms

2° We'll need derivatives for this one!

<u>l'Hôpital's Rule</u>

$\lim_{x\to ?} \frac{f(x)}{g(x)}$ and either $\begin{matrix}\to 0\\ \to 0\end{matrix}$ or $\begin{matrix}\to \pm\infty\\ \to\pm\infty\end{matrix}$

then $\lim_{x\to ?} \frac{f'(x)}{g'(x)}$ (assuming <u>that</u> limit exists)

$$\lim_{x\to\infty} \frac{x}{e^x} \begin{matrix}\to\infty\\ \to\infty\end{matrix} = \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{1}{e^x} \begin{matrix}\to 1\\ \to\infty\end{matrix} = 0.$$