Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2019
Solutions to the Final Examination for Section A

Space-time: *Gym – 19:00-22:00 on Saturday, 14 December, 2019.*

Instructions: Do parts **X** and **Y**, and, if you wish, part **Z**. Please show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Any calculator; (all sides of) one aid sheet; one cranial neural net.

Part X. Do *all four* (4) of **1**–**4**. *[Subtotal = 74]*

1. Compute $\frac{dy}{dx}$ as best you can in any *four* (4) of **f**–**a**. *[20 = 4 × 5 each]*

$$\textbf{a } y = \frac{x^2}{x+1} \qquad \textbf{b. } y = \int_0^{\cos(x)} t^2\,dt \qquad \textbf{c. } y = \int_0^1 13y^{17\pi-1}\,dy$$

$$\textbf{d. } y = \left(e^{x+1}\right)^3 \qquad \textbf{e. } y = (x+1)\ln(x) \qquad \textbf{f. } y = \arctan\left(\sqrt{x}\right)$$

SOLUTIONS. **a.** *Quotient Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{x+1}\right) = \frac{\left[\frac{d}{dx}x^2\right](x+1) - x^2 \cdot \frac{d}{dx}(x+1)}{(x+1)^2} = \frac{2x(x+1) - x^2 \cdot 1}{(x+1)^2}$$
$$= \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2} \qquad \square$$

b. *Fundamental Theorem of Calculus and the Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^{\cos(x)} t^2\,dt = \cos^2(x)\frac{d}{dx}\cos(x) = -\cos^2(x)\sin(x) \qquad \square$$

c. *Trick question.* A definite integral evaluates out to a number, that is, to a constant, and the derivative of any constant is ...

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^1 13y^{17\pi-1}\,dy = 0 \qquad \square$$

d. *Power Rule and Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x+1}\right)^3 = 3\left(e^{x+1}\right)^2\frac{d}{dx}e^{x+1} = 3\left(e^{x+1}\right)^2 e^{x+1}\frac{d}{dx}(x+1)$$
$$= 3\left(e^{x+1}\right)^3 \cdot 1 = 3\left(e^{x+1}\right)^3 = 3e^{3x+3} \qquad \square$$

e. *Product Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x+1)\ln(x)\right] = \left[\frac{d}{dx}(x+1)\right]\ln(x) + (x+1)\left[\frac{d}{dx}\ln(x)\right]$$
$$= 1 \cdot \ln(x) + (x+1) \cdot \frac{1}{x} = \ln(x) + \frac{x}{x} + \frac{1}{x} = \ln(x) + 1 + \frac{1}{x} \qquad \square$$

f. *Chain Rule.*

$$\frac{dy}{dx} = \frac{d}{dx}\arctan\left(\sqrt{x}\right) = \frac{1}{1+\left(\sqrt{x}\right)^2} \cdot \frac{d}{dx}\sqrt{x} = \frac{1}{1+x} \cdot \frac{d}{dx}x^{1/2}$$
$$= \frac{1}{1+x} \cdot \frac{1}{2}x^{-1/2} = \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2(1+x)\sqrt{x}} \qquad \blacksquare$$

2. Evaluate any *four* (4) of the integrals **f**–**a**. *[20 = 4 × 5 each]*

a. $\int 2^{3x}\,dx$ **b.** $\int \frac{1+t}{1+t^2}\,dt$ **c.** $\int z\ln\left(z^2\right)\,dz$

d. $\int_1^e \frac{\ln(w)}{2w}\,dw$ **e.** $\int \frac{2\arctan^2(2v)}{1+4v^2}\,dv$ **f.** $\int_2^e \left(2u + e^u\right)\,du$

Solutions. **a.** *Substitution.* We will substitute $u = 3x$, so $du = 3\,dx$ and $dx = \frac{1}{3}\,du$.

$$\int 2^{3x}\,dx = \int 2^u \frac{1}{3}\,du = \frac{1}{3} \cdot \frac{2^u}{\ln(2)} + C = \frac{2^{3x}}{3\ln(2)} + C \qquad \square$$

b. *Algebra and Substitution.* After a bit of algebra, we will use the substitution $u = t^2 + 1$, so $du = 2t\,dt$ and $t\,dt = \frac{1}{2}\,du$.

$$\int \frac{1+t}{1+t^2}\,dt = \int \frac{1}{1+t^2}\,dt + \int \frac{t}{1+t^2}\,dt = \arctan(t) + \int \frac{1}{u} \cdot \frac{1}{2}\,du$$
$$= \arctan(t) + \frac{1}{2}\ln(u) + C = \arctan(t) + \frac{1}{2}\ln\left(t^2+1\right) + C \qquad \square$$

c. *Integration by Parts.* We will let $u = \ln\left(z^2\right)$ and $v' = z$, so $u' = \frac{1}{z^2}\frac{d}{dz}z^2 = \frac{1}{z^2} \cdot 2z = \frac{2}{z}$ and $v = \frac{z^2}{2}$.

$$\int z\ln\left(z^2\right)\,dz = \frac{z^2}{2}\ln\left(z^2\right) - \int \frac{2}{z} \cdot \frac{z^2}{2}\,dz = \frac{z^2}{2}\ln\left(z^2\right) - \int z\,dz$$
$$= \frac{z^2}{2}\ln\left(z^2\right) - \frac{z^2}{2} + C = \frac{z^2}{2}\left(\ln\left(z^2\right) - 1\right) + C \qquad \square$$

d. *Substitution.* We will substitute $u = \ln(w)$, so $du = \frac{1}{w}\,dw$. We will also change the limits of integration as we go along: $\begin{matrix} w & 1 & e \\ u & 0 & 1 \end{matrix}$.

$$\int_1^e \frac{\ln(w)}{2w}\,dw = \int_0^1 \frac{u}{2}\,du = \frac{1}{2} \cdot \frac{u^2}{2}\Big|_0^1 = \frac{1^2}{4} - \frac{0^2}{4} = \frac{1}{4} - 0 = \frac{1}{4} \qquad \square$$

e. *Substitution.* We will substitute "whole hog": Let $u = \arctan(2v)$, so $\dfrac{du}{dv} = \dfrac{1}{1+(2v)^2} \cdot \dfrac{d}{dv}(2v) = \dfrac{2}{1+4v^2}$, and thus $du = \dfrac{2}{1+4v^2}\, dv$.

$$\int \frac{2\arctan^2(2v)}{1+4v^2}\, dv = \int u^2\, du = \frac{u^3}{3} + C = \frac{1}{3}\arctan^3(2v) + C \qquad \square$$

f. *Basic Properties and Power Rule.*

$$\int_2^e (2u + e^u)\, du = \left(u^2 + e^u\right)\Big|_2^e = \left(e^2 + e^e\right) - \left(2^2 + e^2\right) = e^2 - 2^2 = e^e - 4 \qquad \blacksquare$$

3. Do any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. Find the area of the the region between $y = x^3$ and $y = x$ for $-1 \le x \le 1$.

b. Compute $\lim\limits_{x \to -\infty} xe^x$.

c. Find the equation of the tangent line to $y = e^{x-1}$ at $x = 1$.

d. Use the limit definition of the derivative to show that $\dfrac{d}{dx}\left(x^2+1\right) = 2x$ for all x.

e. Find the maximum value of $f(x) = x + \cos(x)$ on the interval $[0, \pi]$.

f. Use the ε–δ definition of limits to verify that $\lim\limits_{x \to 2}(4 - x) = 2$.

SOLUTIONS. **a.** First, we need to find where the graphs intersect on the interval $[-1, 1]$. $x^3 = x$ exactly when $x = 0$ or $x^2 = 1$, *i.e.* exactly when $x = -1$, 1, or 0.

Second, we need to check which curve is above the other on each of $(-1, 0)$ and $(0, 1)$, which we accomplish by testing the points $x = \pm\frac{1}{2}$. $\left(-\frac{1}{2}\right)^3 = -\frac{1}{8} > -\frac{1}{2}$, so $y = x$ is below $y = x^3$ on $(-1, 0)$, and $\left(\frac{1}{2}\right)^3 = \frac{1}{8} < \frac{1}{2}$, so $y = x$ is above $y = x^3$ on $(0, 1)$.

It follows that the area of the region between $y = x^3$ and $y = x$ for $-1 \le x \le 1$ is given by:

$$\begin{aligned} A &= \int_{-1}^{1} (\text{upper} - \text{lower})\, dx = \int_{-1}^{0} \left(x^3 - x\right)\, dx + \int_0^1 \left(x - x^3\right)\, dx \\ &= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Bigg|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{2}\right)\Bigg|_0^1 = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) - 0 \\ &= -\left(-\frac{1}{4}\right) + \frac{1}{4} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \qquad \square \end{aligned}$$

b. Note that $e^x \to 0$ as $x \to -\infty$, so $\lim\limits_{x \to -\infty} xe^x$ is in an intederminate form. We rewrite it so as to be able to apply l'Hôpital's Rule.

$$\begin{aligned} \lim_{x \to -\infty} xe^x &= \lim_{x \to -\infty} \frac{x \ \to -\infty}{e^{-x} \to +\infty} = \lim_{x \to -\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{-x}} \\ &= \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \lim_{x \to -\infty} -e^x = -0 = 0 \qquad \square \end{aligned}$$

c. When $x = 1$, $y = e^{1-1} = e^0 = 1$, so the tangent line passes through the point $(1,1)$. Since $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{x-1} = e^{x-1}\dfrac{d}{dx}(x-1) = e^{x-1}$, the tangent line has slope $m = \left.\frac{dy}{dx}\right|_{x=1} = e^{1-1} = e^0 = 1$. The tangent line thus has the form $y = x + b$ for some b; since it passes through $(1,1)$, we have $1 = 1 + b$, so $b = 0$. Hence the equation of the tangent line to $y = e^{x-1}$ at $x = 1$ has equation $y = x$. $\square$

d. Recall that the limit definition of the derivative is $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. We apply this to $f(x) = x^2 + 1$:

$$\begin{aligned}\frac{d}{dx}\left(x^2+1\right) &= \lim_{h \to 0} \frac{\left[(x+h)^2+1\right] - \left[x^2+1\right]}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x + 0 = 2x \qquad \square\end{aligned}$$

e. At the end points of the interval we have $f(0) = 0 + \cos(0) = 0 + 1 = 1$ and $f(\pi) = \pi + \cos(\pi) = \pi - 1$, respectively. It remains to find any critical points in the interval and check the value of $f(x)$ at those points. $f'(x) = \frac{d}{dx}\left(x + \cos(x)\right) = 1 - \sin(x)$, which $= 0$ exactly when $\sin(x) = 1$, *i.e.* exactly when $x = n\pi + \frac{\pi}{2}$ for some integer n. There is only one such point in $[0, \pi]$, namely $x = \frac{\pi}{2}$. $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$.

Since π is a little larger than 3, $\frac{\pi}{2}$ is a little larger than 1.5 and $\pi - 1$ is a little larger than 2. Thus $f(0) < f\left(\frac{\pi}{2}\right) < f(\pi)$, and so the maximum value of $f(x) = x + \cos(x)$ on the interval $[0, \pi]$ is $f(\pi) = \pi - 1$. $\square$

f. To verify that $\lim_{x \to 2}(4 - x) = 2$ we need to show that given any $\varepsilon > 0$, there is some $\delta > 0$, such that whenever $|x - 2| < \delta$, we have $|(4 - x) - 2| < \varepsilon$. As usual, we try to reverse engineer a suitable δ from the desired conclusion:

$$\begin{aligned}|(4-x)-2| < \varepsilon &\iff |2-x| < \varepsilon \\ &\iff |x-2| < \varepsilon\end{aligned}$$

Hence taking $\delta = \varepsilon$ works in this case: if $|x - 2| < \delta$, we have $|x - 2| < \varepsilon$ and, running the implications above in reverse, it follows that $|(4 - x) - 2| < \varepsilon$, as required. Thus $\lim_{x \to 2}(4 - x) = 2$ according to the ε–δ definition of limits. $\blacksquare$

4. Find the domain and any and all intercepts, vertical and horizontal asymptotes, intervals of increase, decrease and concavity, and maximum, minimum, and inflection points of $f(x) = \left(\dfrac{x+1}{x}\right)^2 = \dfrac{x^2+2x+1}{x^2}$, and sketch its graph. *[14]*

Solution. We run through the checklist:

i. (Domain) $f(x) = \left(\dfrac{x+1}{x}\right)^2$ is defined exactly when $\dfrac{x+1}{x}$ is defined, which is to say exactly when $x \neq 0$. The domain of $f(x)$ is therefore $\mathbb{R} \setminus \{0\} = \{\, x \in \mathbb{R} \mid x \neq 0 \,\}$.

ii. (Intercepts) Since $f(x)$ is not defined at $x = 0$, it has no y-intercept. As $\left(\frac{x+1}{x}\right)^2 = 0$ exactly when $x + 1 = 0$, $f(x)$ has an x-intercept at $x = -1$.

iii. (Vertical Asymptotes) Since the function is defined and continuous everywhere except at $x = 0$, we take the limit of $f(x)$ at 0 from both directions to see what happens.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{x+1}{x}\right)^2 = \lim_{x \to 0^-} \frac{(x+1)^2}{x^2} \begin{matrix} \to 1 \\ \to 0^+ \end{matrix} = +\infty$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{x+1}{x}\right)^2 = \lim_{x \to 0^+} \frac{(x+1)^2}{x^2} \begin{matrix} \to 1 \\ \to 0^+ \end{matrix} = +\infty$$

It follows that $f(x)$ has a vertical asymptote at $x = 0$; $f(x) \to +\infty$ as $x \to 0$ from either direction.

iv. (Horizontal Asymptotes) We take the limits of $f(x)$ as $x \to \pm\infty$ to check for any horizontal asymptotes:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2 + 2x + 1}{x^2} = \lim_{x \to -\infty} 1 + \frac{2}{x} + \frac{1}{x^2} = 1 + 0^- + 0^+ = 1$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x^2 + 2x + 1}{x^2} = \lim_{x \to +\infty} 1 + \frac{2}{x} + \frac{1}{x^2} = 1 + 0^+ + 0^+ = 1$$

Thus $f(x)$ has $y = 1$ as a horizontal asymptote in both directions.

v. (Increase/Decrease & Max/Min) We compute $f'(x)$ first:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x+1}{x}\right)^2 = 2\left(\frac{x+1}{x}\right) \cdot \frac{d}{dx}\left(\frac{x+1}{x}\right) \\ &= 2\left(\frac{x+1}{x}\right) \cdot \frac{\left[\frac{d}{dx}(x+1)\right] \cdot x - (x+1) \cdot \left[\frac{d}{dx}x\right]}{x^2} \\ &= 2\left(\frac{x+1}{x}\right) \cdot \frac{1 \cdot x - (x+1) \cdot 1}{x^2} = 2\left(\frac{x+1}{x}\right) \cdot \frac{-1}{x^2} = \frac{-2(x+1)}{x^3} \end{aligned}$$

$f'(x)$ is obviously $= 0$ when $x + 1 = 0$, *i.e.* when $x = -1$, and is undefined at $x = 0$. When $x < -1$, then both $x + 1 < 0$ and $x^3 < 0$, so $f'(x) = \frac{-2(x+1)}{x^3} < 0$, which means that $f(x)$ is decreasing. Similarly, when $-1 < x < 0$, then $x + 1 > 0$ but $x^3 < 0$, so $f'(x) = \frac{-2(x+1)}{x^3} > 0$, which means that $f(x)$ is increasing. Note that it follows that $f(x)$ has a minimum at $x = -1$. Finally, when $x < 0$, then $x + 1 > 0$ and $x^3 > 0$, so $f'(x) = \frac{-2(x+1)}{x^3} < 0$, which means that $f(x)$ is increasing. We summarize this in a table:

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, +\infty)$
$f'(x)$	$-$	0	$+$	undef	$-$
$f(x)$	$\downarrow$	min	$\uparrow$	undef	$\downarrow$

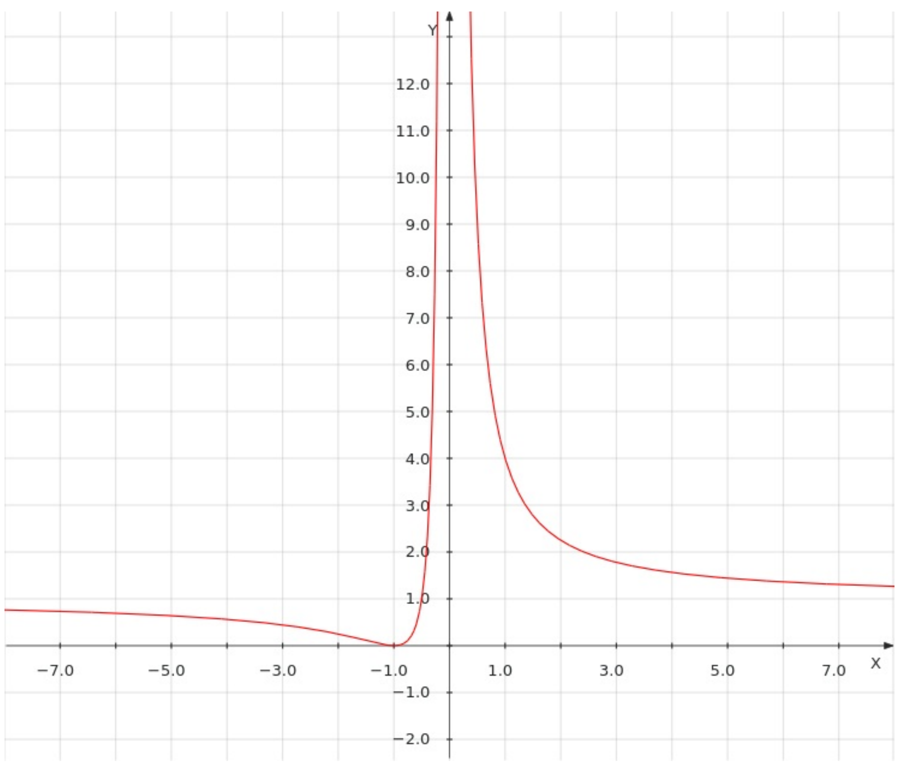
vi. (Concavity/Curvature & Inflection) We compute $f''(x)$ first:

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx} \frac{-2(x+1)}{x^3} = -2 \cdot \frac{\left[\frac{d}{dx}(x+1)\right] \cdot x^3 - (x+1) \cdot \left[\frac{d}{dx} x^3\right]}{\left(x^3\right)^2}$$
$$= -2 \cdot \frac{1 \cdot x^3 - (x+1) \cdot 3x^2}{x^6} = -2 \cdot \frac{x^3 - 3x^3 - 3x^2}{x^6} = -2 \cdot \frac{-2x^3 - 3x^2}{x^6}$$
$$= -2 \cdot \frac{-x^2\,(2x+3)}{x^2 \cdot x^4} = \frac{2(2x+3)}{x^4}$$

$f''(x)$ is obviously undefined at $x = 0$ and $= 0$ when $2x + 3 = 0$, *i.e.* when $x = -\frac{3}{2}$, and is undefined at $x = 0$. Since $x^4 > 0$ whenever $x \neq 0$, $f''(x) = \frac{2(2x+3)}{x^4}$ is > 0, so $f(x)$ is concave up, exactly when $2x + 3 > 0$, *i.e.* when $x > -\frac{3}{2}$. Similarly, $f''(x) = \frac{2(2x+3)}{x^4}$ is < 0, so $f(x)$ is concave down, exactly when $2x + 3 < 0$, *i.e.* when $x < -\frac{3}{2}$. It follows that $f(x)$ has an inflection point at $x = \frac{3}{2}$. We summarize this in a table:

x	$\left(-\infty, -\frac{3}{2}\right)$	$-\frac{3}{2}$	$\left(-\frac{3}{2}, 0\right)$	0	$(0, +\infty)$
$f''(x)$	$-$	0	$+$	undef	$+$
$f(x)$	$\frown$	inflection	$\smile$	undef	$\smile$

vii. (Graph) Cheating slightly, here is the graph of $f(x) = \left(\frac{x+1}{x}\right)^2$:



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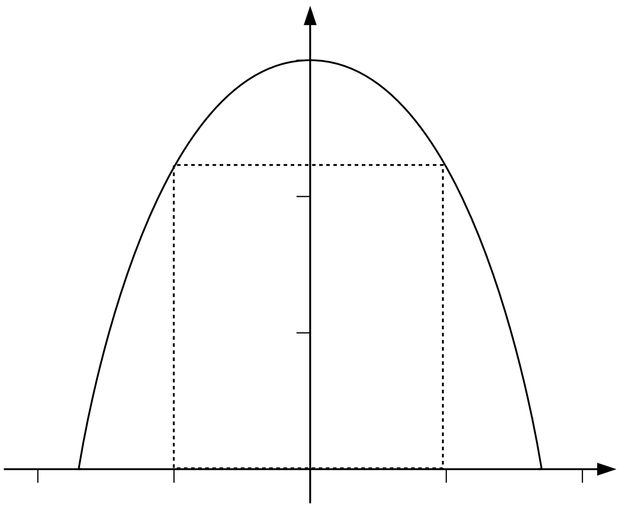
Part Y. Do any *two* (2) of **5**–**7**. *[Subtotal = 26 = 2 × 13 each]*

5. Determine the maximum possible area of a rectangle with two corners on the x-axis and the other two corners on the part of the parabola $y = 3 - x^2$ that is above the x-axis.

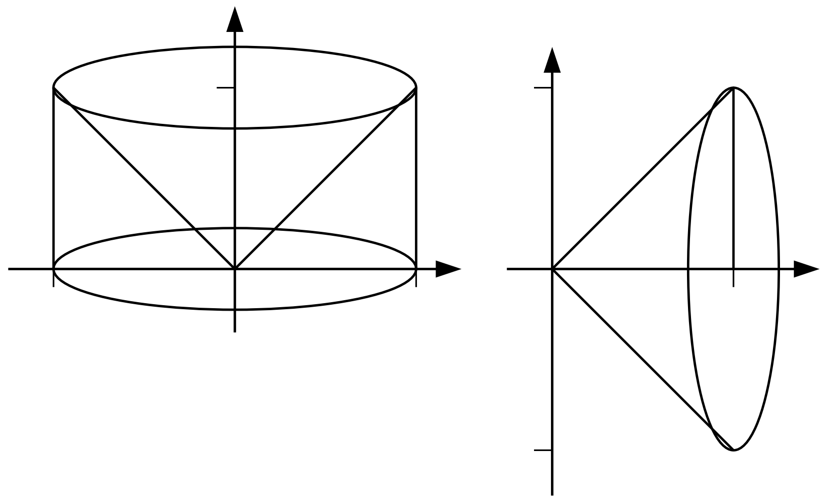


SOLUTION. If the upper right-hand corner of the rectangle has coordinates (x, y), the rectangle has width $x - (-x) = 2x$ and height $y - 0 = y = 3 - x^2$, and thus area $A(x) = 2xy = 2x\left(3 - x^2\right) = 6x - 2x^3$. Since both width and height must be non-negative, we must have $2x \geq 0$ and $3 - x^2 \geq 0$, so $0 \leq x \leq \sqrt{3}$. Note that $A(0) = 6 \cdot 0 - 2 \cdot 0^3 = 0$ and $A\left(\sqrt{3}\right) = 6\sqrt{3} - 2\left(\sqrt{3}\right)^3 = 6\sqrt{3} - 2 \cdot 3\sqrt{3} = 0$. We also need to check any critical points in the interval $\left[0, \sqrt{3}\right]$.

$$A'(x) = \frac{d}{dx}\left(6x - 2x^3\right) = 6 - 6x^2 = 0 \iff x^2 = 1 \iff x = \pm 1$$

Of the two critical points, only $x = +1$ is in $\left[0, \sqrt{3}\right]$. $A(1) = 6 \cdot 1 - 2 \cdot 1^3 = 6 - 2 = 4$, which is greater than $A(0) = 0 = A\left(\sqrt{3}\right)$. Thus the maximum possible area of a rectangle with two corners on the x-axis and the other two corners on the part of the parabola $y = 3 - x^2$ that is above the x-axis is 4. ■

6. Find the volume of the solid obtained by revolving the triangle with vertices at $(0,0)$, $(1,0)$, and $(1,1)$ about, respectively, **a.** the y-axis *[6]* and **b.** the x-axis *[7]*. Sketch each of the solids.



SOLUTION. Crude sketches of the two solids are given above. For both **a** and **b**, note that the base of the right triangle being revolved is part of the line $y = 0$, *i.e.* the x-axis, the side joining $(1,0)$ to $(1,1)$ is part of the line $x = 1$, and the hypotenuse is part of the line $y = x$. Note also that $0 \le x \le 1$ and $0 \le y \le 1$ for this triangle.

We'll compute the volume of **b** first, since it is a slightly simpler object.

b. *(Disks/Washers)* We will use x as the variable because the disks are perpendicular to the axis of revolution, which is part of the x-axis. Since the top edge of the triangle is a piece of $y = x$ and the base of the triangle being revolved is part of the x-axis, the radius of the disk at x is $r = y - 0 = x = 0$. It follows that volume of the solid is given by:

$$V = \int_0^1 \pi r^2\,dx = \int_0^1 \pi x^2\,dx = \left.\frac{\pi x^3}{3}\right|_0^1 = \frac{\pi \cdot 1^3}{3} - \frac{\pi \cdot 0^3}{3} = \frac{\pi}{3} - 0 = \frac{\pi}{3} \qquad \square$$

b. *(Cylindrical Shells)* We will use y as the variable because the cylindrical shells are centered on the axis of revolution, namely the x-axis, and hence are perpendicular to the y-axis. The shell at y has radius $r = y - 0 = y$ and height (length?) $h = 1 - x = 1 - y$, since the cross-section of the original region at y runs from $y = x$ on the left to $x = 1$ on the right. It follows that volume of the solid is given by:

$$V = \int_0^1 2\pi r h\,dy = \int_0^1 2\pi y(1-y)\,dy = \int_0^1 2\pi\left(y - y^2\right)\,dy = 2\pi\left.\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\right|_0^1$$
$$= 2\pi\left(\frac{1^2}{2} - \frac{1^3}{3}\right) - 2\pi\left(\frac{0^2}{2} - \frac{0^3}{3}\right) = 2\pi \cdot \frac{1}{6} - 2\pi \cdot 0 = \frac{\pi}{3} \qquad \square$$

b. *(Geometry)* The solid of revolution is a cone with radius 1 and height 1, so it has volume $V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} \cdot 1^2 \cdot 1 = \frac{\pi}{3}$. $\square$

a. *(Disks/Washers)* We will use y as the variable because the washers are perpendicular to the axis of revolution, which is the y-axis in this case. The outer radius of the washer at y is given by the line $x = 1$, so $R = 1 - 0 = 1$, and the inner radius is given by the line $y = x$, so $r = x - 0 = x = y$. It follows that the volume of the solid is given by:

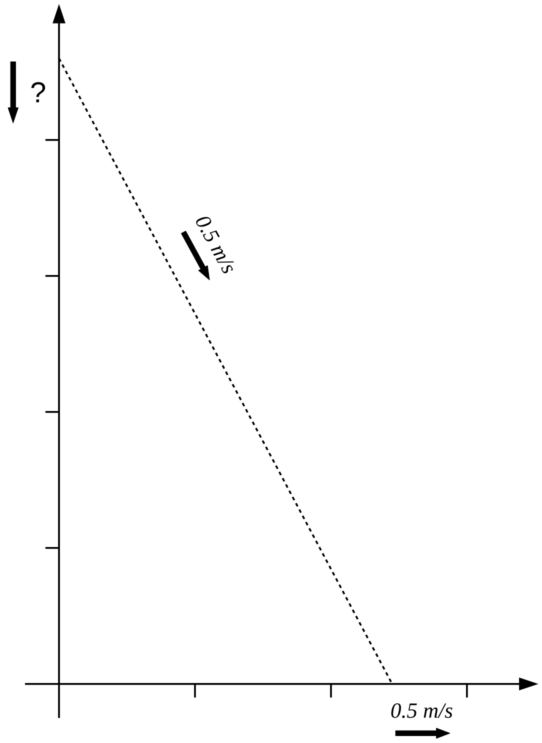
$$V = \int_0^1 \pi \left(R^2 - r^2\right) dy = \int_0^1 \pi \left(1^2 - y^2\right) dy = \pi \left(y - \frac{y^3}{3}\right)\Big|_0^1$$
$$= \pi \left(1 - \frac{1^3}{3}\right) - \pi \left(0 - \frac{0^3}{3}\right) = \pi \cdot \frac{2}{3} - \pi \cdot 0 = \frac{2\pi}{3} \qquad \square$$

a. *(Cylindrical Shells)* We will use x as the variable because the cylindrical shells are centered on the axis of revolution, namely the y-axis, and hence are perpendicular to the x-axis. The cylindrical shell at x has radius $r = x - 0 = x$ and height $h = y - 0 = y = x$, since the cross-section of the original region at x runs from the line $y = x$ down to $y = 0$, *i.e.* the x-axis. It follows that volume of the solid is given by:

$$V = \int_0^1 2\pi r h \, dx = \int_0^1 2\pi x \cdot x \, dx = \int_0^1 2\pi x^2 \, dx = 2\pi \frac{x^3}{3}\Big|_0^1$$
$$= 2\pi \cdot \frac{1^3}{3} - 2\pi \cdot \frac{0^3}{3} = \frac{2\pi}{3} - 0 = \frac{2\pi}{3} \qquad \square$$

a. *(Geometry)* This solid of revolution is a cylinder of height and radius 1 with a cone of height and radius 1 cut out of it. It therefore has volume $V = \pi r^2 h - \frac{\pi}{3} r^2 h = \pi \cdot 1^2 \cdot 1 - \frac{\pi}{3} \cdot 1^2 \cdot 1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. ■

7. The top of a 6 m extension ladder rests against a vertical wall with its base resting on the horizontal floor 2 m from the wall. Something knocks the base of the ladder loose and it begins to slide away from the wall at a constant rate of 0.5 m/s. The knock also loosens the latch keeping the ladder extended and the ladder begins to shorten at a rate of 0.5 m/s. The top of the ladder maintains contact with the wall throughout, but begins to slide down the wall. How fast is the top of the ladder sliding down the wall after 2 s?



SOLUTION. Denote the height of the top of the ladder above the floor by y and the distance of the base of the ladder from the wall by x. With indescribably terrible and terribly indescribable originality, denote the length of the ladder by z. Note that at any given instant, the wall, floor, and ladder form a right triangle with short sides x and y and hypotenuse z, so $x^2 + y^2 = z^2$. We are told that $\frac{dx}{dt} = 0.5\ m/s$ and $\frac{dz}{dt} = -0.5\ m/s$, and that initially $z = 6\ m$ and $x = 2\ m$. We need to find $\frac{dy}{dt}$ after 2 s.

First, note that:

$$x^2 + y^2 = z^2 \implies \frac{d}{dt}x^2 + \frac{d}{dt}y^2 = \frac{d}{dt}z^2 \implies \frac{dx^2}{dx} \cdot \frac{dx}{dt} + \frac{dy^2}{dy} \cdot \frac{dy}{dt} = \frac{dz^2}{dz} \cdot \frac{dz}{dt}$$

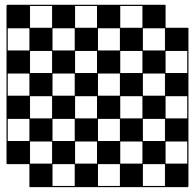
$$\implies 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \implies \frac{dy}{dt} = \frac{2z\frac{dz}{dt} - 2x\frac{dx}{dt}}{2y} = \frac{z\frac{dz}{dt} - x\frac{dx}{dt}}{y}$$

Second, after 2 s we have $x = 2 + 0.5 \cdot 2 = 3\ m$ and $z = 6 - 0.5 \cdot 2 = 5\ m$. It follows that after 2 s. we have $y = \sqrt{z^2 - x^2} = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4\ m$. It follows that $\left.\frac{dy}{dt}\right|_{t=2} = \frac{5 \cdot (-0.5) - 3 \cdot 0.5}{4} = \frac{-8 \cdot 0.5}{4} = -\frac{4}{4} = -1\ m/s$. This means that the top of the ladder is sliding down the wall – note the negative sign! – at a rate of 1 m/s. ■

[Total = 100]

Part Z. Bonus problems! If you feel like it and have the time, do one or both of these.

0. A standard 8×8 chessboard has two opposite corner squares removed. Each of a set of dominos is a rectangle that is exactly the size of two adjacent squares of the board. Show how to lay such dominos down on the board such that each domino covers two adjacent squares of the mutilated board and so that no part of the board is left uncovered, or show why it is impossible to cover the board with dominos in such a way. *[1]*



SOLUTION. *You're on your own!* If you want to look it up, it was stolen from one of Martin Gardner's early *Mathematical Games* columns in *Scientific American*. You'll probably kick yourself if you look it up after not getting it yourself … ■

00. Write a haiku touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

SOLUTION. Write your own, darn it! :-) ■

ENJOY THE BREAK!

(This exam brought to you by Trent University, the Department of Mathematics, and their minions.)