TRENT UNIVERSITY, FALL 2019

MATH 1110H (Section A) Test

Wednesday, 30 October

Time: 15:00–15:50
Space: TSC 1.22

Name: Solutions

STUDENT NUMBER: 0314159

Question	Mark
1	____
2	____
3	____
Total	____ /30

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and (all sides of) an aid sheet.

1. Compute $\frac{dy}{dx}$ for any *three* (3) of parts **a**–**f**. *[12 = 3 × 4 each]*

a. $y = \left(x^2+1\right)^{41}$ **b.** $y = \frac{x^2-1}{x^2+1}$ **c.** $y = 2^{-x}$

d. $y = \frac{\sin(x)}{\tan(x)}$ **e.** $y = \cos\left(x^3\right)$ **f.** $e^{x+y} = 1$

Solutions. **a.** *Power and Chain Rules.*

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2+1\right)^{41} = 41\left(x^2+1\right)^{40} \cdot \frac{d}{dx}\left(x^2+1\right) = 41\left(x^2+1\right)^{40} \cdot 2x = 82x\left(x^2+1\right)^{40} \quad \square$$

b. *Quotient and Power Rules.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{\left[\frac{d}{dx}\left(x^2-1\right)\right]\left(x^2+1\right) - \left(x^2-1\right)\left[\frac{d}{dx}\left(x^2+1\right)\right]}{\left(x^2+1\right)^2} \\ &= \frac{[2x]\left(x^2+1\right) - \left(x^2-1\right)[2x]}{\left(x^2+1\right)^2} = \frac{2x^3+2x-2x^3+2x}{\left(x^2+1\right)^2} = \frac{4x}{\left(x^2+1\right)^2} \quad \square\end{aligned}$$

c. *Memorization and Chain Rule.* $\frac{dy}{dx} = \frac{d}{dx}2^{-x} = \ln(2)2^{-x} \cdot \frac{d}{dx}(-x) = -\ln(2)2^{-x}$ $\square$

c. *Less memorization, some algebra, and Chain Rule.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}2^{-x} = \frac{d}{dx}\left(e^{\ln(2)}\right)^{-x} = \frac{d}{dx}e^{-\ln(2)x} = e^{-\ln(2)x} \cdot \frac{d}{dx}\left(-\ln(2)x\right) \\ &= -\ln(2)e^{-\ln(2)x} = -\ln(2)2^{-x} \quad \square\end{aligned}$$

d. *Simplify first.* Since $y = \frac{\sin(x)}{\tan(x)} = \sin(x) \div \left(\frac{\sin(x)}{\cos(x)}\right) = \sin(x) \cdot \frac{\cos(x)}{\sin(x)} = \cos(x)$, we have $\frac{dy}{dx} = \frac{d}{dx}\cos(x) = -\sin(x)$. $\square$

d. *Quotient Rule, simplify later.*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\sin(x)}{\tan(x)}\right) = \frac{\left[\frac{d}{dx}\sin(x)\right]\tan(x) - \sin(x)\left[\frac{d}{dx}\tan(x)\right]}{\tan^2(x)} \\ &= \frac{\cos(x)\tan(x) - \sin(x)\sec^2(x)}{\tan^2(x)} = \frac{\cos(x) \cdot \frac{\sin(x)}{\cos(x)} - \sin(x)\sec^2(x)}{\tan^2(x)} \\ &= \frac{\sin(x) - \sin(x)\sec^2(x)}{\tan^2(x)} = \frac{\sin(x)\left(1-\sec^2(x)\right)}{\tan^2(x)} \\ &= \frac{-\sin(x)\left(\sec^2(x)-1\right)}{\tan^2(x)} = \frac{-\sin(x)\tan^2(x)}{\tan^2(x)} = -\sin(x) \quad \square\end{aligned}$$

e. *Chain and Power Rules.* $\dfrac{dy}{dx} = \dfrac{d}{dx}\cos\left(x^3\right) = -\sin\left(x^3\right) \cdot \dfrac{d}{dx}x^3 = -3x^2\sin\left(x^3\right)$ □

f. *Solve for y first.* $e^{x+y} = 1 \Leftrightarrow x + y = 0 \Leftrightarrow y = -x$, so $\dfrac{dy}{dx} = \dfrac{d}{dx}(-x) = -1$. □

f. *Implicit Differentiation.*

$$e^{x+y} = 1 \implies \frac{d}{dx}e^{x+y} = \frac{d}{dx}1 \implies e^{x+y}\frac{d}{dx}(x+y) = 0 \implies e^{x+y}\left(1 + \frac{dy}{dx}\right) = 0$$
$$\implies 1 + \frac{dy}{dx} = \frac{0}{e^{x+y}} = 0 \implies \frac{dy}{dx} = -1$$

Note that $e^{x+y} > 0$ no matter what (real number) values x and y may have. ■

2. Do any *two* (2) of parts **a**–**d**. *[8 = 2 × 4 each]*

a. Compute $\lim_{t \to 0} \frac{\tan(t)}{t}$.

b. Use the ε–δ definition of limits to verify that $\lim_{x \to 2} (2x - 1) = 3$.

c. Use the limit definition of the derivative to verify that $\frac{d}{dx}(x+1)^2 = 2(x+1)$.

d. Find the equation of the tangent line to $y = e^{2x}$ at $x = 0$.

Solutions. **a.** Divide and conquer:

$$\lim_{t \to 0} \frac{\tan(t)}{t} = \lim_{t \to 0} \frac{\frac{\sin(t)}{\cos(t)}}{t} = \lim_{t \to 0} \frac{\sin(t)}{t\cos(t)} = \lim_{t \to 0} \frac{\sin(t)}{t} \cdot \frac{1}{\cos(t)}$$
$$= \left(\lim_{t \to 0} \frac{\sin(t)}{t}\right) \cdot \left(\lim_{t \to 0} \frac{1}{\cos(t)}\right) = 1 \cdot \frac{1}{\cos(0)} = \frac{1}{1} = 1 \quad \square$$

b. We need to show that given any $\varepsilon > 0$, one can find a $\delta > 0$, such that (for all x) if $|x - 2| < \delta$, then $|(2x - 1) - 3| < \varepsilon$.

Suppose we are given some $\varepsilon > 0$. As usual, we reverse-engineer the corresponding δ from the desired conclusion:

$$|(2x-1) - 3| < \varepsilon \iff |2x - 4| < \varepsilon \iff 2\,|x - 2| < \varepsilon \iff |x - 2| < \frac{\varepsilon}{2}$$

If we take $\delta = \frac{\varepsilon}{2}$, then whenever $|x - 2| < \delta = \frac{\varepsilon}{2}$, we get $|(2x - 1) - 3| < \varepsilon$ by following the (fully-reversible!) reasoning above from right to left.

It follows by the ε–δ definition of limits that $\lim_{x \to 2} (2x - 1) = 3$. $\square$

c. By the limit definition of the derivative:

$$\frac{d}{dx}(x+1)^2 = \lim_{h \to 0} \frac{((x+h)+1)^2 - (x+1)^2}{h}$$
$$= \lim_{h \to 0} \frac{\left(x^2 + xh + x \cdot 1 + hx + h^2 + h \cdot 1 + \cdot x + 1 \cdot h + 1^2\right) - \left(x^2 + 2x + 1\right)}{h}$$
$$= \lim_{h \to 0} \frac{2hx + 2h}{h} = \lim_{h \to 0} (2x + 2) = 2x + 2 = 2(x+1) \quad \square$$

d. When $x = 0$, $y = e^{2 \cdot 0} = e^0 = 1$, so the tangent line passes through the point $(0, 1)$, which means that it has a y-intercept of $b = 1$.

Since $\frac{dy}{dx} = \frac{d}{dx} e^{2x} = e^{2x} \cdot \frac{d}{dx}(2x) = 2e^{2x}$, the slope of the tangent line at $x = 0$ is $m = \left.\frac{dy}{dx}\right|_{x=0} = 2e^{2 \cdot 0} = 2e^0 = 2 \cdot 1 = 2$.

Thus the equation of the tangent line to $y = e^{2x}$ at $x = 0$ is $y = mx + b = 2x + 1$. ■

3. Find the domain and any and all intercepts, asymptotes, intervals of increase and decrease, maximum and minimum points, intervals of curvature, and inflection points of the function $f(x) = \frac{1}{\sqrt{x^2+1}} = \left(x^2+1\right)^{-1/2}$, and sketch its graph. *[10]*

Solution. We run through the given checklist:

i. Domain. Since $x^2+1>0$ for all $x \in \mathbb{R}$, $f(x) = \frac{1}{\sqrt{x^2+1}}$ is defined for all x too. Note that since $f(x)$ is a composition of continuous functions, it is continuous wherever it is defined, which is to say it is continuous everywhere.

ii. Intercepts. Since $f(0) = \frac{1}{\sqrt{0^2+1}} = \frac{1}{\sqrt{1}} = \frac{1}{1} = 1$, the y-intercept is 1. On the other hand, since $f(x) = \frac{1}{\sqrt{x^2+1}} > 0$ for all x, it does not have any x-intercept.

iii. Asymptotes. Since, as noted above, $f(x) = \frac{1}{\sqrt{x^2+1}}$ is defined and continuous for all x, it cannot have any vertical asymptotes. We compute the usual limits to find any horizontal asymptotes; note that $\sqrt{x^2+1} \to +\infty$ as $x \to -\infty$ and as $x \to +\infty$:

$$\lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}} \begin{matrix} \to 1 \\ \to +\infty \end{matrix} = 0^+$$
$$\lim_{x\to+\infty} \frac{1}{\sqrt{x^2+1}} \begin{matrix} \to 1 \\ \to +\infty \end{matrix} = 0^+$$

It follows that $y = f(x)$ has a horizontal asymptote of $y = 0$, which it approaches from above, in both directions.

iv. Intervals of increase and decrease, and maximum and minimum points.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^2+1\right)^{-1/2} = -\frac{1}{2}\left(x^2+1\right)^{-3/2} \cdot \frac{d}{dx}\left(x^2+1\right) = -\frac{1}{2}\left(x^2+1\right)^{-3/2} \cdot (2x) \\ &= -x\left(x^2+1\right)^{-3/2} = \frac{-x}{\left(\sqrt{x^2+1}\right)^3} \end{aligned}$$

Since $x^2+1>0$, and hence also $\left(x^2+1\right)^{-3/2} > 0$, for all x, $f'(x) = 0$, > 0, or < 0, respectively, exactly when $-x = 0$, > 0, or < 0, respectively, *i.e.* exactly when $x = 0$, < 0, or > 0, respectively. Since $f'(x) > 0$ when $x < 0$, $f(x)$ is increasing for $x < 0$, and $f'(x) < 0$ when $x > 0$, so $f(x)$ is decreasing for $x > 0$, and so $f(x)$ has a maximum at $x = 0$. We summarize this information in a table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$+$	0	$-$
$f(x)$	$\uparrow$	max	$\downarrow$

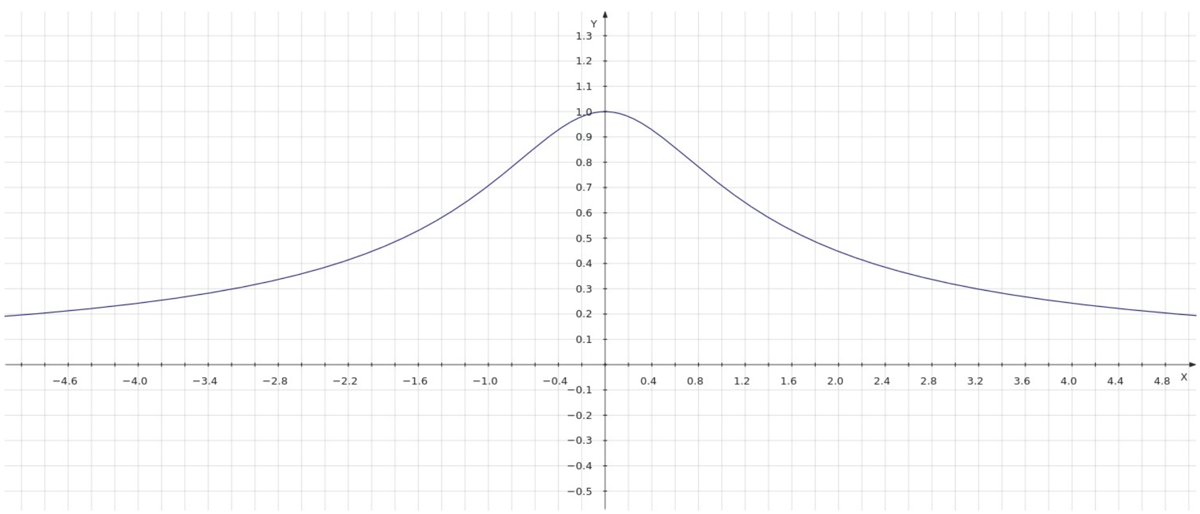
v. Intervals of curvature and points of inflection.

$$f''(x) = \frac{d}{dx}\left(-x\left(x^2+1\right)^{-3/2}\right) = \left[\frac{d}{dx}(-x)\right]\cdot\left(x^2+1\right)^{-3/2} + (-x)\cdot\left[\frac{d}{dx}\left(x^2+1\right)^{-3/2}\right]$$
$$= -1\cdot\left(x^2+1\right)^{-3/2} + (-x)\cdot\left(-\frac{3}{2}\right)\left(x^2+1\right)^{-5/2}\cdot\left[\frac{d}{dx}\left(x^2+1\right)\right]$$
$$= -\left(x^2+1\right)^{-3/2} + x\cdot\frac{3}{2}\left(x^2+1\right)^{-5/2}\cdot(2x)$$
$$= -\left(x^2+1\right)\left(x^2+1\right)^{-5/2} + 3x^2\left(x^2+1\right)^{-5/2} = \left(-x^2-1+3x^2\right)\left(x^2+1\right)^{-5/2}$$
$$= \left(2x^2-1\right)\left(x^2+1\right)^{-5/2} = \frac{2x^2-1}{\left(x^2+1\right)^{5/2}} = \frac{2x^2-1}{\left(\sqrt{x^2+1}\right)^5}$$

Since $x^2+1>0$, and hence also $\left(x^2+1\right)^{-5/2}>0$, for all x, $f'(x)=0$, >0, or <0, respectively, exactly when $2x^2-1=0$, >0, or <0, respectively, *i.e.* exactly when $x=\pm\frac{1}{\sqrt{2}}$, $|x|>\frac{1}{\sqrt{2}}$, or $|x|<\frac{1}{\sqrt{2}}$, respectively. It follows that $f(x)$ is concave up when $x<-\frac{1}{\sqrt{2}}$ and when $x>\frac{1}{\sqrt{2}}$, and concave down when $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$, so it has inflection points when $x=\pm\frac{1}{\sqrt{2}}$. We summarize this information in a table:

x	$\left(-\infty,-\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{\sqrt{2}}$	$\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}},\infty\right)$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$	infl	$\frown$	infl	$\smile$

vi. The graph. It's cheating, but it's way more convenient to have a computer do the work. In this case, it's a graphing program called `kmplot`.



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