Mathematics 1110H – Calculus I: Limits, Derivatives, and Integrals

TRENT UNIVERSITY, Fall 2018

Solutions to Assignment #8

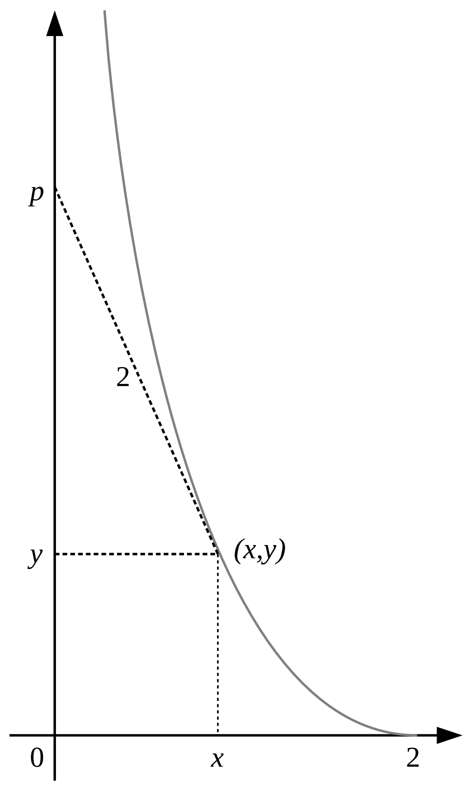
Max and Min and Hare today ...

Little Max is walking Big Min the math dog on a 2 m leash, moving left along the x-axis in the Cartesian plane. Big Min is an eager walker, keeping the leash fully extended at all times. Just as Big Min reaches the origin, Big Min spots a Splitting Hare on the y-axis. The Splitting Hare runs straight up the y-axis to get away and Big Min follows, also running directly up the y-axis while dragging Little Max, who continues to hang onto the leash. Thanks to the leash, Little Max is always moving directly towards Big Min and is always 2 m away. Dragging Little Max along slows Big Min just enough so as to never catch up to the Splitting Hare.

1. Find the function $f(x)$ whose graph $y = f(x)$ for $0 < x \leq 2$ is the path traced out by Little Max while being dragged by Big Min when following Splitting Hare. *[10]*

Hint: Your real task is to set up a suitable differential equation. Once you have that, get a certain computer program to solve it for you ...

SOLUTION. To set up the differential equation, consider the following sketch of the situation after a little while, with Little Max at the point (x, y) and Big Min at the point $(0, p)$ on the y-axis.



Recall that the leash is 2 m long and pulls Max directly towards Min. This last means that the leash is part of the tangent line to the curve Max is being dragged along, the slope of which is given by the derivative $\frac{dy}{dx}$ of the function whose graph is the curve. On the other hand, if we compute this slope as rise over run from Min's position at $(0, p)$ to Max's position at (x, y), *i.e.* from one end of the leash to the other, we get $\frac{y-p}{x-0} = \frac{y-p}{x}$. It follows that $\frac{dy}{dx} = \frac{y-p}{x}$. This isn't quite the differential equation we're looking for yet: we need to express p in terms of x and/or y.

Note that the triangle whose vertices are the points $(0, y)$, $(0, p)$, and $x, y)$ is a right triangle with short sides of lengths $x - 0 = x$ and $p - y$ and a hypotenuse of length 2. By the Pythagorean Theorem, it follows that $x^2 + (p-y)^2 = 2^2$, which we can rewrite a little as $x^2 + (y-p)^2 = 4$. (We're doing this because $\frac{dy}{dx} = \frac{y-p}{x}$ has $y - p$ instead of $p - y$ and get away with it because $(p-y)^2 = (y-p)^2$.) Solving for $y - p$ now gives us $y - p = -\sqrt{4-x^2}$ – we take the negative square root because $p > y$, so $y - p < 0$.

It follows that $\frac{dy}{dx} = \frac{y-p}{x} = -\frac{\sqrt{4-x^2}}{x}$. This is our differential equation, which we need to solve together with the given initial condition that $y(2) = 0$. (Recall that Max is at $x = 2$ on the x-axis when Min starts chasing the rabbit.) We use `Maple`'s `dsolve` command to do so:

```
> dsolve( { diff(y(x)) = -sqrt(4-x^2)/x, y(2)=0 }, y(x) )
```

$$y(x) = -\sqrt{4-x^2} + 2\,\mathrm{arctanh}\left(\frac{2}{\sqrt{4-x^2}}\right) + \mathrm{I}\,\pi$$

Unfortunately, this solution has a problem in that the "I" is `Maple`'s way of presenting the square root of -1, usually denoted by i. Using arctanh in the solution is also a bit unusual when compared to what usually comes out in hand-worked solutions, although inverse hyperbolic functions often come up in those. Such solutions usually end up looking something like the following:

$$y(x) = -\sqrt{4-x^2} + 2\mathrm{arcsech}\left(\frac{x}{2}\right) = -\sqrt{4-x^2} + 2\ln\left(\frac{2+\sqrt{4-x^2}}{x}\right)$$

You'll be able to work these out by hand once you learn how to handle more advanced intergration techniques in MATH 1120H. :-) ■