**Mathematics 1101Y – Calculus I: Functions and calculus of one variable**
Trent University, 2012–2013

**Solutions to Assignment #1**
**Plotting with `Maple`†**

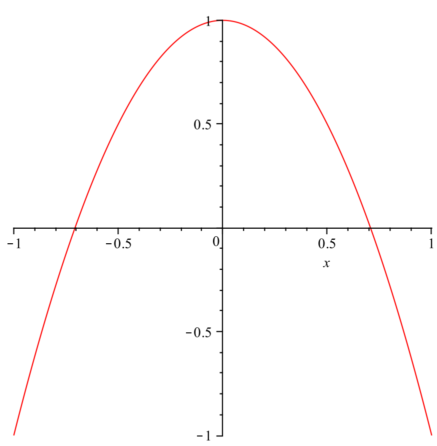
Before attempting the questions below, please read through Chapter 1, for the basics of graphing various functions in Cartesian coordinates, and through §11.1 and §11.3 of Chapter 11, for the basics of parametric curves and polar coordinates, respectively. (The basic definitions of how parametric curves and polar coordinates work are embedded in this assignment for your convenience, but you might want some additional explanations and examples.) You should also read the handout *A very quick start with* `Maple` and play around with `Maple` a little. It might also be useful to skim though *Getting started with* `Maple 10` by Gilberto E. Urroz – read those parts concerned with plotting curves more closely! – and perhaps keep it handy as a reference. You can find links to both documents on the MATH 1101Y web page. `Maple`'s help facility may also come in handy, especially when trying to make out the intricacies of what the `plot` command and its options and variations do. Finally, make use of the `Maple` labs!

A curve is easy to graph, at least in principle, if it can be described by a function of $x$ in Cartesian coordinates.

**1.** Use `Maple` to plot the curves defined by $y = 1 - 2x^2$, $y = 1$, $y = \sqrt{1 - x^2}$, and $y = \sqrt{|x|}$, respectively, for $-1 \le x \le 1$ in each case. [Please submit a printout of your worksheet(s).] *[2]*

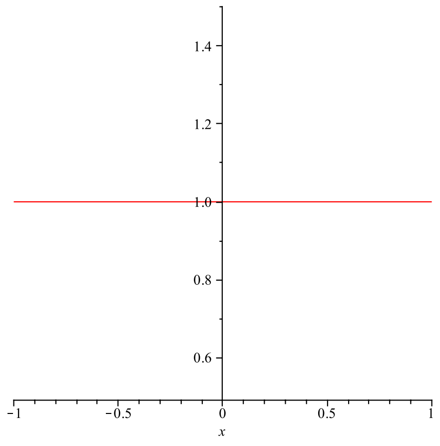
Solution. Suitable instances of the `plot` command and their output are given below. [The graphs have been reduced in size to save some space.] To enable the use of the `implicitplot` and `polarplot` commands in later questions, your instructor began his worksheet by loading the `plots` package.

```
> with(plots):
> plot(1-2*x^2,x=-1..1)
```

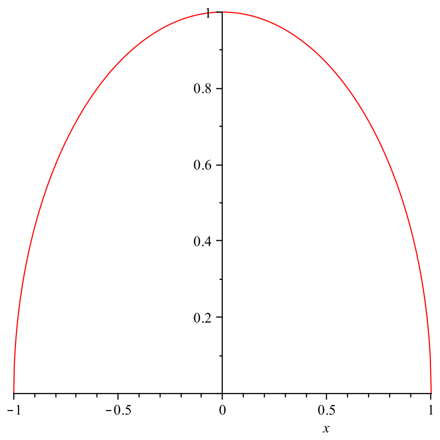



† You may well feel that `Maple` is plotting *against* you ...

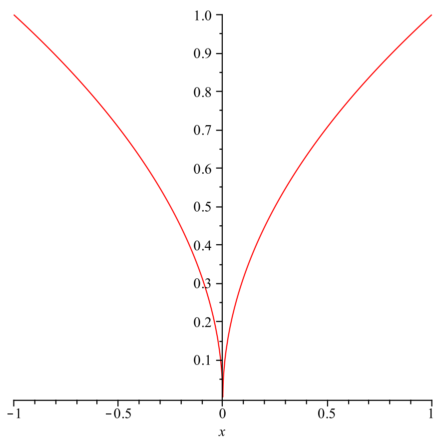
```
> plot(1,x=-1..1)
```




```
> plot(sqrt(1-x^2),x=-1..1)
```




```
> plot(sqrt(abs(x)),x=-1..1)
```



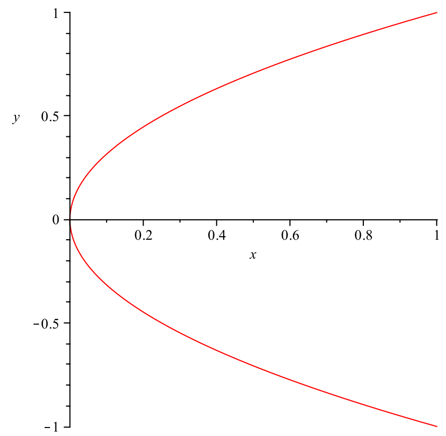

■

In many cases, a curve is difficult to break up into pieces that are defined by functions of $x$ (or of $y$) and so is defined implicitly by an equation relating $x$ and $y$; that is, the curve consists of all points $(x, y)$ such that $x$ and $y$ satisfy the equation.

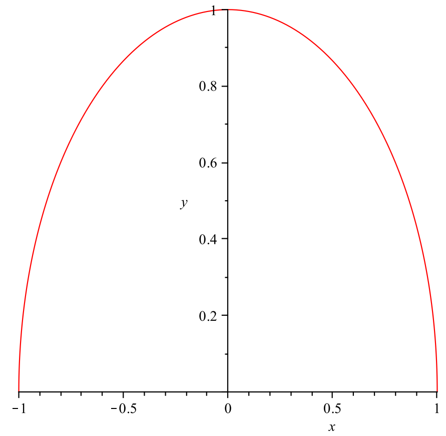
**2.** Use `Maple` to plot the curves implicitly defined by $x = y^2$, for $-1 \leq y \leq 1$, $x^2 + y^2 = 1$ for $y \geq 0$, $\left(x^2 + y^2\right)^3 = 4x^2y^2$, and $|x| + |y| = 1$, respectively, the latter two for all $x$ and $y$ satisfying each equation. [Please submit a printout of your worksheet(s).] *[2]*

Solution. Suitable instances of the `implicitplot` command and their output are given below. [The graphs have been reduced in size to save some space.] Recall that your instructor began his worksheet by loading the `plots` package to enable the use of the `implicitplot`. Note the use of the `gridrefine=2` option to get smoother curves.

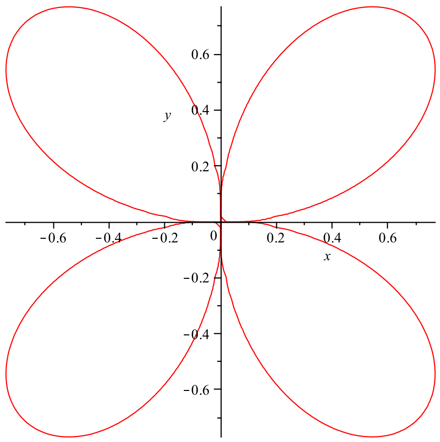
```
> implicitplot(x=y^2,y=-1..1,gridrefine=2)
```



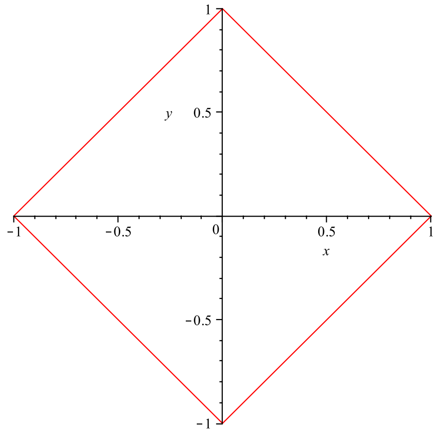

```
> implicitplot(x^2+y^2=1,x=-1..1,y=0..1,gridrefine=2)
```



> `implicitplot((x^2+y^2)^3=4*x^2*y^2,x=-1..1,y=0..1,gridrefine=2)`




> `implicitplot(abs(x)+abs(y)=1,x=-1..1,y=0..1,gridrefine=2)`



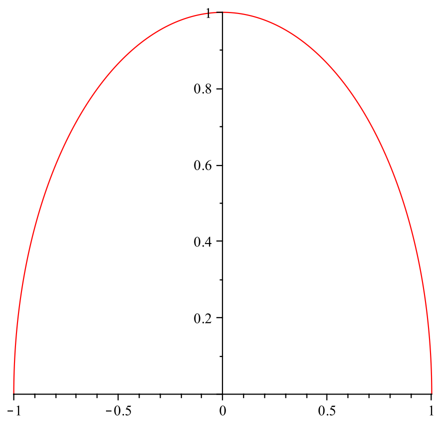

■

Another way to describe or define a curve in two dimensions is by way of *parametric equations,* $x = f(t)$ and $y = g(t)$, where the $x$ and $y$ coordinates of points on the curve are simultaneously specified by plugging a third variable, called the *parameter* (in this case $t$), into functions $f(t)$ and $g(t)$. This approach can come in handy for situations where it is impossible to describe all of a curve as the graph of a function of $x$ (or of $y$) and arises pretty naturally in various physics problems. (Think of specifying, say, the position $(x, y)$ of a moving particle at time $t$.)

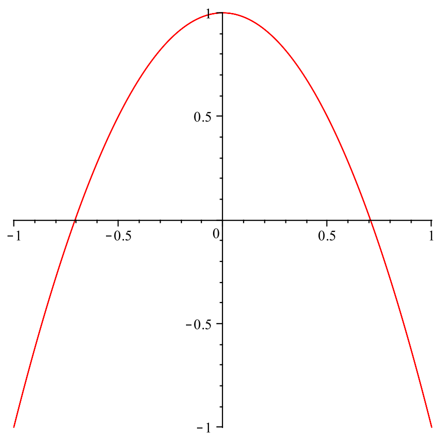
**3.** Use `Maple` to plot the parametric curves given by $x = \cos(t)$ and $y = \sin(t)$ for $0 \leq t \leq \pi$, $x = \sin(t)$ and $y = \cos(2t)$, and $x = 2\sin(t)\cos^2(t)$ and $y = 2\sin^2(t)\cos(t)$, respectively, these two curves for $0 \leq t \leq 2\pi$. [Please submit a printout of your worksheet(s).] *[1.5]*

Solution. Suitable instances of the `plot` command and their output are given below. [The graphs have been reduced in size to save some space.]

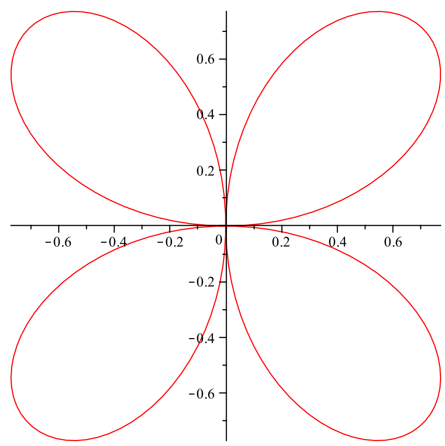
> `plot([cos(t),sin(t),t=0..Pi])`




> `plot([sin(t),cos(2*t),t=0..2*Pi])`




> `plot([2*sin(t)*(cos(t))^2,2*(sin(t))^2*cos(t),t=0..2*Pi])`



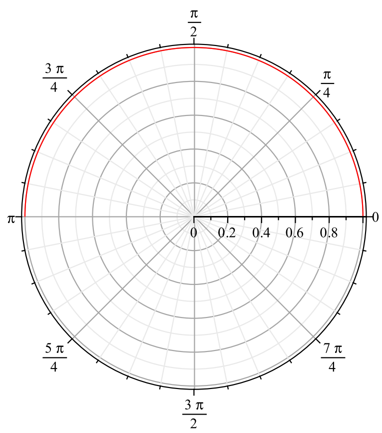

∎

Polar coordinates are an alternative to the usual two-dimensional Cartesian coordinate. The idea is to locate a point by its distance $r$ from the origin and its direction, which is given by the (counterclockwise) angle $\theta$ between the positive $x$-axis and the line from the origin to the point. Thus, if $(r, \theta)$ are the polar coordinates of some point, then its Cartesian coordinates are given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$. (Note that for purposes of calculus it is usually more convenient to measure angles in radians rather than degrees.) Polar coordinates come in particularly handy when dealing with curves that wind around the origin, since such curves can often be conveniently represented by an equation of the form $r = f(\theta)$ for some functon $\theta$. If $r$ is negative for a given $\theta$, we interpret that as a distance of $|r|$ in the opposite direction, *i.e.* the direction $\pi + \theta$.

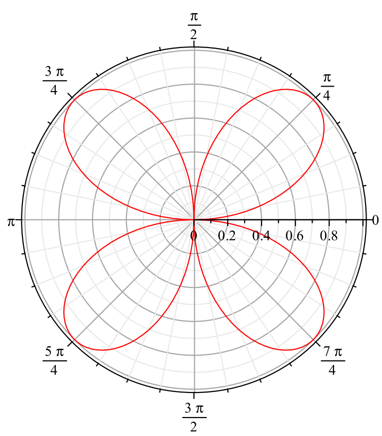
**4.** Use `Maple` to plot the curves in polar coordinates given by $r = 1$ for $0 \leq \theta \leq \pi$, $r = \sin(2\theta)$, and $r = \csc(\theta)$, respectively, these two curves for $0 \leq \theta \leq 2\pi$. [Please submit a printout of your worksheet(s).] *[1.5]*

Solution. Suitable instances of the `polarplot` command and their output are given below. [The graphs have been reduced in size to save some space.] Recall that your instructor began his worksheet by loading the `plots` package to enable the use of the `polarplot`. One could also get these curves by using the `coords=polar` option in the `plot` command; an example if this is given for the second curve.

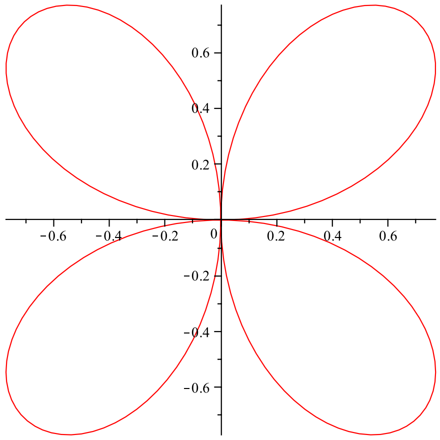
> `polarplot(1,theta=0..Pi)`



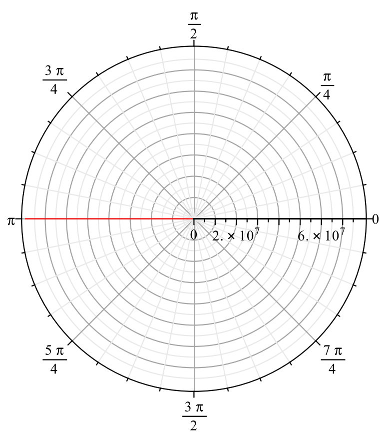

> `polarplot(sin(2*theta),theta=0..2*Pi)`



> `plot(sin(2*z),z=0..2*Pi,coords=polar)`




> `polarplot(csc(theta),theta=0..2*Pi)`




Note the rather large distance scale in the last plot. The line in question is not actually the $x$-axis, though it is awfully close on the scale in the plot. ■

**5.** Some of the curves in problems **1**–**4** are actually the same curve. (With different presentations ... ) Which ones are the same? *[2]*

Solution. Interpreting "the same" as "exactly the same set of points in the Euclidean plane," the following curves are the same:

- $y = 1 - 2x^2$, for $-1 \leq x \leq 1$, from **1**, and
  $x = sin(t)$ and $y = \cos(2t)$, for $0 \leq t \leq 2\pi$, from **3**.
- $y = \sqrt{1 - x^2}$, for $-1 \leq x \leq 1$, from **1**, and
  $x^2 + y^2 = 1$, for $y \geq 0$, from **2**, and
  $x = \cos(t)$ and $y = \sin(t)$, for $0 \leq t \leq \pi$, from **3**, and
  $r = 1$, for $0 \leq \theta \leq \pi$, from **4**.

- $x^2 + y^2 = 1$ for $y \geq 0$ [$-1 < x < 1$ and $-1 < y < 1$ for all of it], from **2**, and $x = 2\sin(t)\cos^2(t)$ and $y = 2\sin^2(t)\cos(t)$, for $0 \leq t \leq 2\pi$, from **3**, and $r = \sin(2\theta)$, for $0 \leq \theta \leq 2\pi$, from **4**.

There are various others that are similar in shape or, indeed, overlap (*e.g.* $y = \sqrt{|x|}$ from **1** and $x = y^2$ from **2**), but they are not exactly the same. ■

**6.** Find a representation of the form $r = f(\theta)$ in polar coordinates for the curve whose equation in Cartesian coordinates is $y = 1 - 2x^2$, as best you can. [That is, you need to figure out what the function $f(\theta)$ ought to be.] *[1]*

Solution. We plug the relations $x = r\cos(theta)$ and $y = r\sin(\theta)$ into $y = 1 - 2x^2$ and try to solve for $r$:

$$\begin{aligned} y = 1 - 2x^2 \quad &\Longrightarrow \quad r\sin(\theta) = 1 - 2\left[r\cos(\theta)\right]^2 = 1 - 2r^2\cos^2(\theta) \\ &\Longrightarrow \quad 2\cos^2(\theta)r^2 + \sin(\theta)r - 1 = 0 \end{aligned}$$

At this point we have a quadratic equation in $r$, some of whose coefficients are functions of $\theta$. Applying the quadratic equation gives:

$$\begin{aligned} r &= \frac{-\sin(\theta) \pm \sqrt{\sin^2(\theta) - 4 \cdot 2\cos^2(\theta) \cdot (-1)}}{2 \cdot 2\cos^2(\theta)} = \frac{-\sin(\theta) \pm \sqrt{[1 - \cos^2(\theta)] + 8\cos^2(\theta)}}{4\cos^2(\theta)} \\ &= \frac{-\sin(\theta) \pm \sqrt{1 + 7\cos^2(\theta)}}{4\cos^2(\theta)} = -\frac{1}{4}\tan(\theta)\sec(\theta) \pm \frac{1}{4}\sec^2(\theta)\sqrt{1 + 7\cos^2(\theta)} \end{aligned}$$

Note that the last two forms of the answer are completely equivalent and that neither is really essentially simpler than the other. The multiplicity of trig functions and identities means that there are a *lot* of ways one could rewrite the answer. The *gung-ho* can also try to figure out for which $\theta$ the expression in the answer actually makes sense. ■

References


1. *A very quick start with* `Maple`, by Stefan Bilaniuk, which can found (pdf) at: `http://euclid.trentu.ca/math/sb/1101Y/2012-2013/MATH1101Y-maple-start.pdf`
2. *Calculus: Early Transcendentals* (2nd Edition), by Jon Rogawski, W.H. Freeman, New York, 2012, ISBN-10: 1-4292-6009-2, ISBN-13: 978-1-4292-6009-1.
3. *Getting started with Maple 10*, by Gilberto E. Urroz (2005), which can found (pdf) at: `http://euclid.trentu.ca/math/sb/1101Y/2012-2013/GettingStartedMaple10.pdf`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