Math 1100 — Calculus, Quiz #18A — 2010-04-05

Are the following series absolutely convergent, conditionally convergent, or divergent? Justify your answer in each case.

(25) 1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$.

Solution: This series is $\boxed{\text{divergent.}}$ The Integral Test says that the series converges if and only if the improper integral $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,\mathrm{d}x$ converges. But

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,\mathrm{d}x \underset{(*)}{=} \int_{\ln(2)}^{\infty} \frac{1}{\sqrt{u}}\,\mathrm{d}u = \lim_{N\to\infty} 2u^{1/2}\Big|_{u=\ln(2)}^{u=N} = \lim_{N\to\infty}\left(2\sqrt{N} - 2\sqrt{\ln(2)}\right) = \infty.$$

Thus, the integral is divergent, and thus, so is the series. Here $(*)$ is the change of variables $u := \ln(x)$ so that $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$. □

(25) 2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,3^n\,n^2}{n!}$.

Solution: This series is $\boxed{\text{absolutely convergent}}$. To see this, we use the Ratio Test. Let $a_n := \dfrac{(-1)^n 3^n n^2}{n!}$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{3^{n+1}(n+1)^2/(n+1)!}{3^n n^2/n!} = \frac{3(n+1)^2\cdot n!}{n^2\cdot(n+1)!} = \frac{3(n+1)^2}{n^2\cdot(n+1)} = \frac{3(n+1)}{n^2}.$$

Thus, $$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{3(n+1)}{n^2} = 0 < 1.$$

Thus, the Ratio Test says the series is absolutely convergent. □

(25) 3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^5)}{n^{3/2}}$.

Solution: This series is $\boxed{\text{absolutely convergent}}$. To see this, observe that $|\sin(n^5)| \leq 1$ for all $n \in \mathbb{N}$. Thus, $\left|\dfrac{\sin(n^5)}{n^{3/2}}\right| \leq \dfrac{1}{n^{3/2}}$. But the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges (it is a p-series with $p = 3/2 > 1$). Thus, the Comparison Test tells us that the series $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\sin(n^5)}{n^{3/2}}\right|$ also converges. □

(25) 4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+5}}$.

Solution: This series is $\boxed{\text{conditionally convergent}}$ but *not* absolutely convergent. To see this, first observe that the sequence $\left\{\frac{1}{\sqrt{n^2+5}}\right\}_{n=1}^{\infty}$ is decreasing (because the function $f(x) = \sqrt{x^2+5}$ is increasing). Also,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+5}} \quad = \quad 0.$$

Thus, the Alternating Series Test says that the series converges. However, the series does not converge absolutely. To see this, we use the Limit Comparison Test to compare the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+5}}$ to the divergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{1/n}{1/\sqrt{n^2+5}} \quad = \quad \lim_{n\to\infty} \frac{\sqrt{n^2+5}}{n} \quad = \quad \lim_{n\to\infty} \sqrt{1+5/n^2} \quad = \quad 1 \quad \neq \quad 0.$$

Thus, as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+5}}$ also diverges. □