Math 1100 — Calculus, HW #3 — Due Friday, March 5, 2012

L^2 space and Fourier theory

Solutions

'Common mistakes' are indicated in your marked assignment with circled numbers, e.g. ①, ②, ③, etc. These labels are explained in the remarks following the solutions to each question.

1. Let $f : [0, \pi] \longrightarrow \mathbb{R}$ and $g : [0, \pi] \longrightarrow \mathbb{R}$ be two integrable functions. The *inner product*[1] of f and g is defined:

$$\langle f, g \rangle \quad := \quad \int_0^\pi f(x)\, g(x)\, dx. \tag{1}$$

We say that f is *orthogonal* to g if $\langle f, g \rangle = 0$. The L^2-*norm* of f is defined:[2]

$$\|f\|_2 \quad := \quad \sqrt{\langle f, f \rangle} \quad = \quad \sqrt{\int_0^\pi f(x)^2\, dx}. \tag{2}$$

The set of all functions with finite L^2-norm is called L^2 *space*. It is very important in quantum mechanics, and in the study of random processes.

$(\frac{25}{200})$ (a) For all $n \in \mathbb{N}$, define $\mathbf{S}_n(x) := \sin(n\,x)$ for all $x \in [0, \pi]$. Thus, $\langle \mathbf{S}_n, \mathbf{S}_m \rangle = \int_0^\pi \sin(n\,x)\, \sin(m\,x)\, dx$. Show that $\mathbf{S}_n$ is orthogonal to $\mathbf{S}_m$ whenever $n \neq m$. (**Hint:** *Use the identity:* $\sin(a) \cdot \sin(b) \;=\; -\frac{1}{2}\Big(\cos(a+b) - \cos(a-b)\Big)$.)

Solution:

$$\begin{aligned}
\langle \mathbf{S}_n, \mathbf{S}_m \rangle &= \int_{\mathbb{X}} \mathbf{S}_n(x) \cdot \mathbf{S}_m(x)\, dx \quad = \quad \int_0^\pi \sin(nx) \cdot \sin(mx)\, dx \\
&= -\frac{1}{2} \int_0^\pi \cos(nx + mx) - \cos(nx - mx)\, dx \\
&= -\frac{1}{2} \left(\int_0^\pi \cos\left[(n+m)x\right]\, dx \; - \; \int_0^\pi \cos\left[(n-m)x\right]\, dx \right) \\
&= -\frac{1}{2} \left(\frac{1}{n+m} \sin\left[(n+m)x\right] \Big|_{x=0}^{x=\pi} \; - \; \frac{1}{n-m} \sin\left[(n-m)x\right] \Big|_{x=0}^{x=\pi} \right) \\
&= -\frac{1}{2} \left(\frac{1}{n+m}(0-0) \; - \; \frac{1}{n-m}(0-0) \right) \quad = \quad \boxed{0.}
\end{aligned}$$

□

$(\frac{25}{200})$ (b) Compute $\|\mathbf{S}_n\|_2$ using formula (2). (Your answer should be independent of the value of n).

[1] Compare this to the inner product of two 3-dimensional vectors: $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$.

[2] Compare this to the norm of a 3-dimensional vector: $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$

Solution:

$$\begin{aligned}
\|\mathbf{S}_n\|_2^{\,2} \quad &= \quad \int_{\mathbb{X}} \mathbf{S}_n(x)^2 \; dx \quad = \quad \int_0^\pi \sin(nx)\cdot\sin(nx) \; dx \\
&= \quad -\frac{1}{2}\int_0^\pi \cos(nx+nx) - \cos(nx-nx) \; dx \\
&= \quad -\frac{1}{2}\left(\int_0^\pi \cos(2nx) \; dx \; - \; \int_0^\pi \cos(0) \; dx\right) \\
&= \quad -\frac{1}{2}\left(\frac{1}{2n}\sin(2nx)\Big|_{x=0}^{x=\pi} \; - \; \int_0^\pi 1 \; dx\right) \\
&= \quad -\frac{1}{2}\left(\frac{1}{2n}(0-0) \; - \; \pi\right) \quad = \quad -\frac{1}{2}\,(-\pi) \quad = \quad \frac{\pi}{2}.
\end{aligned}$$

Thus, $\|\mathbf{S}_n\|_2 \;=\; \sqrt{\frac{\pi}{2}}$. □

$(\frac{10}{200})$ (c) For any two functions $f, g : [0,\pi]\longrightarrow\mathbb{R}$, show that $\langle f,g\rangle = \langle g,f\rangle$. (This is called *symmetry*).

Solution: $\langle f,g\rangle = \displaystyle\int_0^\pi f(x)g(x)\; dx = \int_0^\pi g(x)f(x)\; dx = \langle g,f\rangle$. □

$(\frac{10}{200})$ (d) For any four functions $e, f, g, h : [0,\pi]\longrightarrow\mathbb{R}$, show that $\langle e+f, g+h\rangle = \langle e,g\rangle + \langle e,h\rangle + \langle f,g\rangle + \langle f,h\rangle$. (This is called *bilinearity*).

Solution:

$$\begin{aligned}
\langle e+f, g+h\rangle \quad &= \quad \int_0^\pi \Big(e(x)+f(x)\Big)\cdot\Big(g(x)+h(x)\Big)\; dx \\
&= \quad \int_0^\pi e(x)g(x) + e(x)h(x) + f(x)g(x) + f(x)h(x)\; dx \\
&= \quad \int_0^\pi e(x)g(x)\; dx + \int_0^\pi e(x)h(x)\; dx + \int_0^\pi f(x)g(x)\; dx + \int_0^\pi f(x)h(x)\; dx \\
&= \quad \langle e,g\rangle + \langle e,h\rangle + \langle f,g\rangle + \langle f,h\rangle.
\end{aligned}$$

□

$(\frac{10}{200})$ (e) Deduce: if f and g are *orthogonal*, then $\|f+g\|_2^{\,2} = \|f\|_2^{\,2} + \|g\|_2^{\,2}$. (This is called the *Pythagorean identity*).

Solution: We have

$$\begin{aligned}
\|f+g\|_2^{\,2} \;\underset{(\dagger)}{=}\; \langle f+g, f+g\rangle \;&\underset{(*)}{=}\; \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle \\
&\underset{(\diamond)}{=}\; \|f\|_2^{\,2} + 2\langle f,g\rangle + \|g\|_2^{\,2} \;\underset{(\ddagger)}{=}\; \|f\|_2^{\,2} + \|g\|_2^{\,2}.
\end{aligned}$$

Here, $(\dagger)$ is by defining equation (2), $(*)$ is by part (d), $(\diamond)$ is by part (c) and defining equation (2), and $(\ddagger)$ is because $\langle f,g\rangle = 0$ because f is orthogonal to g. □

$(\frac{20}{200})$ (f) The *Cauchy-Bunyakowski-Schwarz inequality* states that $|\langle f,g\rangle| \;\leq\; \|f\|_2\cdot\|g\|_2$ for all functions f and g. The CBS inequality is easy to prove, but we will just assume it here. Using the CBS inequality, prove the *Triangle Inequality*:[3] $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

Solution: We have

$$\begin{aligned}
\|f+g\|_2{}^2 &\underset{(\dagger)}{=} \langle f+g, f+g\rangle \underset{(*)}{=} \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle \\
&\underset{(\diamond)}{=} \|f\|_2{}^2 + 2\langle f,g\rangle + \|g\|_2{}^2 \ \leq\ \|f\|_2{}^2 + 2|\langle f,g\rangle| + \|g\|_2{}^2 \\
&\underset{(\ddagger)}{\leq} \|f\|_2{}^2 + 2\|f\|_2 \cdot \|g\|_2 + \|g\|_2{}^2 \quad = \quad (\|f\|_2 + \|g\|_2)^2.
\end{aligned}$$

Here, $(\dagger)$ is by defining equation (2), $(*)$ is by part (d), $(\diamond)$ is by part (c) and defining equation (2), and $(\ddagger)$ is by the CBS inequality.

Thus, $\|f+g\|_2{}^2 \leq (\|f\|_2 + \|g\|_2)^2$. Taking the square root of both sides of this inequality, we get $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$. □

2. For any $f : [0,\pi] \longrightarrow \mathbb{R}$ and any $n \in \mathbb{N}$, the nth *Fourier coefficient* of f is defined:

$$B_n \quad := \quad \frac{2}{\pi}\langle f, \mathbf{S}_n\rangle \quad = \quad \frac{2}{\pi}\int_0^\pi f(x)\,\sin(nx)\,dx. \tag{3}$$

The *Fourier series*[4] for f is then the function defined by the infinite summation:[5]

$$\sum_{n=1}^{\infty} B_n \,\sin(n\,x). \tag{4}$$

If f is continuously differentiable on $[0,\pi]$, then the Fourier series (4) converges to $f(x)$ for all $x \in (0,\pi)$. [6] Fourier series are enormously important in probability theory, signal processing, and the study of partial differential equations.

$(\frac{25}{200})$ (a) Suppose $f(x) = 1$ for all $x \in [0,\pi]$. Show that, in this case, the nth Fourier coefficient B_n in equation (3) is given by

$$B_n \quad = \quad \begin{cases} 4/n\pi & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Conclude that, for this function, the Fourier series (4) is:

$$\frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots\right) \tag{5}$$

[3] The Triangle Inequality and Symmetry properties together mean that the L^2 norm defines a concept of 'distance' in L^2-space. Thus, L^2-space has a 'geometry', closely analogous to three-dimensional Euclidean space, except that it is infinite-dimensional.

[4] Strictly speaking, this is the Fourier *sine* series for f. One can also define Fourier series using the functions $\cos(nx)$ or $\exp(\mathbf{i}nx)$.

[5] Compare this to expressing a vector in $\mathbb{R}^3$ in terms of an orthogonal coordinate system.

[6] In fact, if we are willing to consider more exotic forms of convergence, then the Fourier series (4) converges to f even if f is an extremely pathological function with many discontinuities.

Solution: We have

$$\begin{aligned} B_n &= \frac{2}{\pi}\int_0^\pi \sin(nx)\ dx \quad = \quad \frac{-2}{n\pi}\cos(nx)\Big|_{x=0}^{x=\pi} \quad = \quad \frac{2}{n\pi}\Big[1-(-1)^n\Big] \\ &= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is } \textbf{odd} \\ 0 & \text{if } n \text{ is } \textbf{even} \end{cases}. \end{aligned}$$

Thus, the Fourier sine series is:

$$\frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n}\sin(nx) \quad = \quad \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots\right)$$

□

$(\frac{25}{200})$ (b) Suppose $f(x) = x^2$ for all $x \in [0,\pi]$. Find a formula for nth Fourier coefficient B_n, as defined by equation (3).

Solution: We will apply integration by parts *twice*.

$$\begin{aligned} \int_0^\pi x^2\cdot\sin(nx)\ dx &\underset{(*)}{=} \frac{-1}{n}\left(x^2\cdot\cos(nx)\Big|_{x=0}^{x=\pi} - 2\int_0^\pi x\cos(nx)\ dx\right) \\ &\underset{(\dagger)}{=} \frac{-1}{n}\left[\pi^2\cdot\cos(n\pi) - \frac{2}{n}\left(x\cdot\sin(nx)\Big|_{x=0}^{x=\pi} - \int_0^\pi \sin(nx)\ dx\right)\right] \\ &= \frac{-1}{n}\left[\pi^2\cdot(-1)^n + \frac{2}{n}\left(\frac{-1}{n}\cos(nx)\Big|_{x=0}^{x=\pi}\right)\right] \\ &= \frac{-1}{n}\left[\pi^2\cdot(-1)^n - \frac{2}{n^2}\Big((-1)^n - 1\Big)\right] \\ &= \frac{2}{n^3}\Big((-1)^n - 1\Big) + \frac{(-1)^{n+1}\pi^2}{n}. \end{aligned}$$

Here $(*)$ is integration by parts with $u := x^2$ and $dv := \sin(nx)\ dx$, so that $du = 2x\ dx$ and $v = -\frac{1}{n}\cos(nx)$. Next, $(\dagger)$ is integration by parts with $u := x$ and $dv := \cos(nx)\ dx$, so that $du = dx$ and $v = \frac{1}{n}\sin(nx)$.

Thus,

$$\begin{aligned} B_n &= \frac{2}{\pi}\int_0^\pi x^2\cdot\sin(nx)\ dx \\ &= \begin{cases} \dfrac{-2\pi}{n} & \text{if } n \text{ is even;} \\[2ex] \dfrac{-4}{\pi n^3} + \dfrac{2\pi}{n} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

□

3. Let $f : \mathbb{R}\longrightarrow\mathbb{R}$ and $g : \mathbb{R}\longrightarrow\mathbb{R}$ be two integrable functions. The *convolution* of f and g is the function $(f * g) : \mathbb{R}\longrightarrow\mathbb{R}$ defined as follows: for every $x \in \mathbb{R}$,

$$(f*g)(x) \quad := \quad \int_{-\infty}^{\infty} f(y)\,g(x-y)\ dy.$$

(We will assume this integral converges). Convolutions arise frequently in probability theory, signal processing, partial differential equations, and Fourier theory.

$\left(\frac{25}{200}\right)$ (a) Suppose $f(x) = \begin{cases} (1/2) & \text{if} \quad -1 \leq x \leq 1; \\ 0 & \text{otherwise.} \end{cases}$

Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be any function. For all $x \in \mathbb{R}$, show that

$$(f * g)(x) \quad = \quad \frac{1}{2} \int_{x-1}^{x+1} g(z)\ dz$$

That is, $(f * g)(x)$ is simply the *average value*[7] of g over the interval $[x-1, x+1]$.

Solution: We have

$$f * g(x) \quad = \quad \int_{-\infty}^{\infty} f(y) g(x-y)\ dy \quad = \quad \frac{1}{2} \int_{-1}^{1} g(x-y)\ dy \quad \underset{(*)}{=} \quad -\frac{1}{2} \int_{x+1}^{x-1} g(z)\ dz$$
$$\underset{(\dagger)}{=} \quad \frac{1}{2} \int_{x-1}^{x+1} g(z)\ dz,$$

as desired. Here, $(*)$ is the change of variables $z := x - y$, so that $dz = -dy$. Next, in $(\dagger)$ we reverse the bounds of integration and multiply by (-1). □

$\left(\frac{25}{200}\right)$ (b) For any functions f and g, show that $f * g = g * f$. (Technically: the convolution operator is *commutative*).

Solution:

$$(g * f)(x) \quad = \quad \int_{-\infty}^{\infty} g(y) \cdot f(x-y)\ dy \quad \underset{(s)}{=} \quad \int_{\infty}^{-\infty} g(x-z) \cdot f(z) \cdot (-1)\ dz$$
$$= \quad \int_{-\infty}^{\infty} f(z) \cdot g(x-z)\ dz \quad = \quad (f * g)(x).$$

Here, step (s) was the substitution $z = x - y$, so that $y = x - z$ and $dy \ = \ -dz$. □

Bonus problem: For any three functions f, g, and h, show that $(f * g) * h = f * (g * h)$. (Technically: the convolution operator is *associative*).

Solution: Fix $x \in \mathbb{R}$. Then

$$\begin{aligned}
f * (g * h)(x) \quad &= \quad \int_{-\infty}^{\infty} f(y) (g * h)(x-y)\ dy \\
&= \quad \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} g(z) h\left[(x-y) - z\right]\ dz \right)\ dy \\
&= \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(z) h\left[x - (y+z)\right]\ dz\ dy \\
&\underset{(*)}{=} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) g(w-y) h(x-w)\ dw\ dy \\
&= \quad \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) g(w-y)\ dy \right) h(x-w)\ dw \\
&= \quad \int_{-\infty}^{\infty} (f * g)(w) \cdot h(x-w)\ dw \\
&= \quad (f * g) * h(x).
\end{aligned}$$

[7] This example is typical: the convolution $f*g$ can often be interpreted as a sort of 'local weighted averaging' of the function g, with f playing the role of the 'weight function'. For example, in image processing, (two-dimensional) convolutions are used to create 'blurring' and 'smudging' effects in images. Conversely, we can 'sharpen' or 'enhance' an image by applying a 'reverse convolution' —but we need advanced Fourier analysis to explain how to do this.

Here, $(*)$ is the change of variables $z := w - y$; hence $w = z + y$ and $dw = dz$. $\square$