MATH 1101 2009
Assignment 2

Due January 22, 2010

5 points for each problem. Show all your work.

1. Find the limit. Use l'Hospital's Rule where appropriate.

(a)

$$\lim_{x\to\infty}(e^x+x)^{\frac{1}{x}}$$

Solution: Let $y=(e^x+x)^{\frac{1}{x}}$. $\ln y=\frac{1}{x}\ln(e^x+x)$

$$\begin{aligned}
\lim_{x\to\infty}\ln y &= \lim_{x\to\infty}\frac{\ln(e^x+x)}{x}(\text{Note: This is in the form }\frac{\infty}{\infty}.)\\
&= \lim_{x\to\infty}\frac{\frac{1}{e^x+x}(e^x+1)}{1}\\
&= \lim_{x\to\infty}\frac{e^x+1}{e^x+x}(\text{Note: This is in the form }\frac{\infty}{\infty}.)\\
&= \lim_{x\to\infty}\frac{e^x}{e^x+1}(\text{Note: This is in the form }\frac{\infty}{\infty}.)\\
&= \lim_{x\to\infty}\frac{e^x}{e^x}=1.
\end{aligned}$$

Therefore

$$\lim_{x\to\infty}y=\lim_{x\to\infty}e^{\ln y}=e.$$

□

(b)

$$\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$$

Solution: Let $y=\left(\frac{2x-3}{2x+5}\right)^{2x+1}$.

$$\begin{aligned}
\lim_{x\to\infty}\ln y &= \lim_{x\to\infty}\ln\left(\frac{2x-3}{2x+5}\right)^{2x+1}\\
&= \lim_{x\to\infty}(2x+1)\ln\left(\frac{2x-3}{2x+5}\right)\\
&= \lim_{x\to\infty}\frac{\ln\left(\frac{2x-3}{2x+5}\right)}{\frac{1}{(2x+1)}}\ (\text{Note: This is in the form }\frac{0}{0}.)\\
&= \lim_{x\to\infty}\frac{\frac{2x+5}{2x-3}\frac{2(2x+5)-2(2x-3)}{(2x+5)^2}}{\frac{-2}{(2x+1)^2}}=\lim_{x\to\infty}\frac{\frac{2x+5}{2x-3}\frac{16}{(2x+5)^2}}{\frac{-2}{(2x+1)^2}}\\
&= \lim_{x\to\infty}-\frac{2x+5}{2x-3}\frac{16}{(2x+5)^2}\frac{(2x+1)^2}{2}\\
&= \lim_{x\to\infty}-\frac{8(2x+1)^2}{(2x-3)(2x+5)}=\lim_{x\to\infty}-\frac{32x^2+32x+8}{4x^2+4x-15}\\
&= -8
\end{aligned}$$

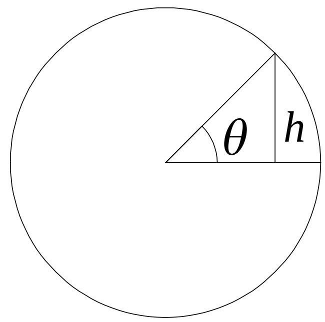
Therefore,

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{-8}.$$

$\square$

2. Page 247, #40. A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?

   Solution:



   As in the figure, we have

$$\begin{aligned} h &= 10\sin\theta \\ \frac{dh}{dt} &= 10\cos\theta\frac{d\theta}{dt} \\ \frac{d\theta}{dt} &= \frac{2\pi}{2} = \pi. \end{aligned}$$

   When the seat is 16m above ground level,

$$\begin{aligned} h &= 16 - 10 = 6 \\ \cos\theta &= \frac{\sqrt{10^2 - 6^2}}{10} = \frac{8}{10} \\ \frac{dh}{dt} &= 10\cos\theta\frac{d\theta}{dt} \\ &= 10 \cdot \frac{8}{10} \cdot \pi \\ &= 8\pi \text{ m/min.} \end{aligned}$$

$\square$

3. Page 286, #24. Suppose that $3 \leq f'(x) \leq 5$ for all values of x. Show that $18 \leq f(8) - f(2) \leq 30$.

   Proof: By the Mean Value Theorem,

$$f(8) - f(2) = f'(c)(8-2) = 6f'(c).$$

   for some c in the interval $(2, 8)$. Since $3 \leq f'(x) \leq 5$, $18 \leq 6f'(c) \leq 30$. Therefore

$$18 \leq f(8) - f(2) \leq 30.$$

$\square$

4. Page 329, #42. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u $(u < v)$, then the time required to swim a distance L is $L/(v-u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v-u}$$

where a is the proportionality constant.

   (a) Determine the value of v that minimizes E.

   (b) Sketch the graph of E.

Note: This result has been verified experimentally: migrating fish swim against a current at a speed 50% greater than the current speed.

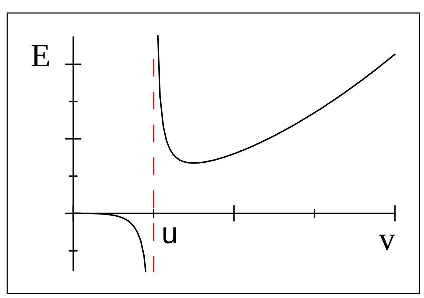
Solution: (a)

$$E'(v) = aL\frac{(v-u)\,3v^2 - v^3}{(v-u)^2}$$

$$\begin{aligned} E' &= 0 \\ &\Leftrightarrow (v-u)\,3v^2 - v^3 = 0 \\ &\Leftrightarrow 3v^3 - 3uv^2 - v^3 = 0 \\ &\Leftrightarrow 2v^3 - 3uv^2 = 0 \\ &\Leftrightarrow v^2(2v - 3u) = 0. \end{aligned}$$

The critical numbers are $v = 0$ and $v = \frac{3}{2}u$. If $v < \frac{3}{2}u$, $2v - 3u < 0$, $E' < 0$. If $v > \frac{3}{2}u$, $2v - 3u > 0$, $E' > 0$. By the First Derivative Test, $v = \frac{3}{2}u$ is a local minimum. Since we must have $v > u$, this also the absolute maximum.

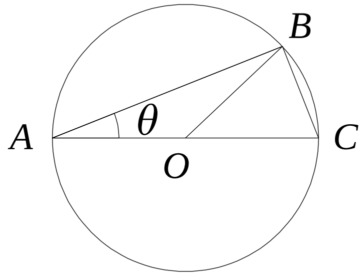
(b)



□

5. Page 329, #46. A woman at a point A on the shore of a circular lake with radius 2 mi wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at the rate of 4 mi/h and row a boat at 2 mi/h. How should she proceed?

Solution:



As in the figure, we assume that she rows from A to B then walks from B to C. Let the angle $\angle BAC$ be θ. The angle $\angle BOC$ would be 2θ. The angle $\angle ABC$ is an right angle. So we have

$$\begin{aligned} \frac{\overline{AB}}{\overline{AC}} &= \frac{\overline{AB}}{4} = \cos\theta \\ \overline{AB} &= 4\cos\theta \end{aligned}$$

The time she spend from A to B is $\frac{4\cos\theta}{2} = 2\cos\theta$. (Note: The length $\overline{AB}$ can also be calculated using the Law of Cosines.) The length of the arc from B to C is $2\,(2\theta) = 4\theta$. The time she will spend on this part is $\frac{4\theta}{4} = \theta$. Therefore, the time she needs to reach C as a function of θ is

$$T(\theta) = 2\cos\theta + \theta,\ 0 \le \theta \le \frac{\pi}{2}.$$

Since

$$T'(\theta) = -2\sin\theta + 1$$

$$\begin{aligned} T'(\theta) &= 0 \Leftrightarrow 2\sin\theta = 1 \\ &\Leftrightarrow \sin\theta = \frac{1}{2}, \end{aligned}$$

$\theta = \frac{\pi}{6}$ is the only critical number. We compare $T(0)$, $T\left(\frac{\pi}{6}\right)$, and $T\left(\frac{\pi}{2}\right)$.

$$\begin{aligned} T(0) &= 2 \\ T\left(\frac{\pi}{6}\right) &= 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6} \approx 2.2556 \\ T\left(\frac{\pi}{2}\right) &= 2\cdot 0 + \frac{\pi}{2} \approx 1.5708 \end{aligned}$$

Since the minimum is achieved at $\theta = \frac{\pi}{2}$, she should walk all the way. It will take 1. 57 hours to reach C. $\square$