Mathematics 110 – Calculus of one variable
Trent University, 2003-2004

§A Test #2 Solutions

1. Compute any *three* of the integrals in parts **a-f**. *[12 = 3 × 4 each]*

a. $\displaystyle\int_0^{\pi/2} \cos^3(x)\,dx$ **b.** $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$ **c.** $\displaystyle\int_2^{\infty} \frac{1}{\sqrt{x}}\,dx$

d. $\displaystyle\int \frac{\arctan(x)}{x^2+1}\,dx$ **e.** $\displaystyle\int \ln\left(x^2\right)\,dx$ **f.** $\displaystyle\int_1^2 \frac{1}{x^2-2x+2}\,dx$

Solutions.

a. Trig identity followed by a substitution:

$$\int_0^{\pi/2} \cos^3(x)\,dx = \int_0^{\pi/2} \cos^2(x)\cos(x)\,dx = \int_0^{\pi/2} \left(1-\sin^2(x)\right)\cos(x)\,dx$$

Letting $u = \sin(x)$, we get $du = \cos(x)\,dx$; note that $u = 0$ when $x = 0$ and $u = 1$ when $x = \pi/2$.

$$= \int_0^1 \left(1-u^2\right)\,du = \left(u - \frac{1}{3}u^3\right)\Big|_0^1$$

$$= \left(1 - \frac{1}{3}1^3\right) - \left(0 - \frac{1}{3}0^3\right) = \frac{2}{3} \quad \blacksquare$$

b. Partial fractions:

$$\int \frac{1}{x^2+3x+2}\,dx = \int \frac{1}{(x+1)(x+2)}\,dx = \int \left(\frac{A}{x+1} + \frac{B}{x+2}\right)\,dx$$

We need to determine A and B:

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)} = \frac{(A+B)x+(2A+B)}{(x+1)(x+2)}$$

Comparing coefficients in the numerators, it follows that $A + B = 0$ and $2A + B = 1$. Subtracting the first equation from the second gives $A = (2A+B)-(A+B) = 1-0 = 1$; substituting this back into the first equation gives $1 + B = 0$, so $B = -1$. We can now return to our integral:

$$\int \frac{1}{x^2+3x+2}\,dx = \int \frac{1}{(x+1)(x+2)}\,dx = \int \left(\frac{1}{x+1} + \frac{-1}{x+2}\right)\,dx$$

$$= \int \frac{1}{x+1}\,dx - \int \frac{1}{x+2}\,dx = \ln(x+1) - \ln(x+2) + C$$

$$= \ln\left(\frac{x+1}{x+2}\right) + C \quad \blacksquare$$

c. Improper integral:

$$\int_2^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_2^t x^{1/2}\,dx$$
$$= \lim_{t\to\infty} \left.\frac{x^{3/2}}{3/2}\right|_2^t = \lim_{t\to\infty}\left(\frac{2}{3}t^{3/2} - \frac{2}{3}2^{3/2}\right) = \infty$$

... because $t^{3/2} > t$ and $t \to \infty$. Hence this improper integral does not converge. ■

d. Substitution:

$$\int \frac{\arctan(x)}{x^2+1}\,dx = \int u\,du \qquad \text{where } u = \arctan(x) \text{ and } du = \frac{1}{x^2+1}\,dx$$
$$= \frac{1}{2}u^2 + C = \frac{1}{2}\arctan^2(x) + C \quad ■$$

e. Integration by parts:

$$\int \ln\left(x^2\right)\,dx = \int 2\ln(x)\,dx \qquad \text{Let } u = \ln(x) \text{ and } v' = 2, \text{ so } u' = \frac{1}{x} \text{ and } v = 2x.$$
$$= \ln(x)\cdot 2x - \int \frac{1}{x}\cdot 2x\,dx = 2x\ln(x) - \int 2\,dx = 2x\ln(x) - 2x + C \quad ■$$

f. Completing the square and substitution:

$$\int_1^2 \frac{1}{x^2-2x+2}\,dx = \int_1^2 \frac{1}{(x^2-2x+1)+1}\,dx = \int_1^2 \frac{1}{(x-1)^2+1}\,dx$$

Let $u = x-1$, then $du = dx$; note that $u = 0$ when $x = 1$ and $u = 1$ when $x = 2$.

$$= \int_0^1 \frac{1}{u^2+1}\,du = \arctan(u)|_0^1$$
$$= \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Note that $\arctan(1) = \frac{\pi}{4}$ and $\arctan(0) = 0$ because $\tan\left(\frac{\pi}{4}\right) = 1$ and $\tan(0) = 0$. ■

2. Do any *two* of parts **a-d**. *[8 = 2 × 4 each]*

a. Find a definite integral computed by the Right-hand Rule sum

$$\lim_{n\to\infty}\sum_{i=0}^{n}\left(1+\frac{i^2}{n^2}\right)\cdot\frac{1}{n}.$$

[The sum should have been $\sum_{i=1}^{n}\cdots$ *instead of* $\sum_{i=0}^{n}\cdots$*. Darn typo!]*

Solution. The general Right-hand Rule formula is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

Comparing the general sum above to the given one reveals that $f\left(a+\frac{b-a}{n}\right) = 1 + \frac{i^2}{n^2}$ and $\frac{b-a}{n} = \frac{1}{n}$. It follows from the latter that $b - a = 1$. If we arbitrarily choose $a = 0$, it will follow that $b = 1$ and $f\left(a + i\frac{b-a}{n}\right) = f\left(0 + \frac{i}{n}\right) = f\left(\frac{i}{n}\right)$. It follows that $f\left(\frac{i}{n}\right) = 1 + \frac{i^2}{n^2} = 1 + \left(\frac{i}{n}\right)^2$, that is, $f(x) = 1 + x^2$.

Plugging all this into the integral side of the Right-hand Rule formula, we see that:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i^2}{n^2}\right) \cdot \frac{1}{n} = \int_0^1 \left(1 + x^2\right)\,dx$$

It is worth noting that we could have chosen a to be any real number. This would, of course, result in a different value of b (since $b - a = 1$) and a different function $f(x)$. ■

b. Compute $\dfrac{d}{dx}\left(\displaystyle\int_0^{\tan(x)} e^{\sqrt{t}}\,dt\right)$.

Solution. This is a job for the Fundamental Theorem of Calculus and the Chain Rule:

$$\begin{aligned}
\frac{d}{dx}\left(\int_0^{\tan(x)} e^{\sqrt{t}}\,dt\right) &= \frac{d}{dx}\left(\int_0^{u} e^{\sqrt{t}}\,dt\right) && \text{where } u = \tan(x) \\
&= \frac{d}{du}\left(\int_0^{u} e^{\sqrt{t}}\,dt\right) \cdot \frac{du}{dx} && \text{by the Chain Rule} \\
&= e^{\sqrt{u}} \cdot \frac{du}{dx} && \text{by the Fundamental Theorem} \\
&= e^{\sqrt{\tan(x)}} \cdot \frac{d}{dx}\tan(x) \\
&= e^{\sqrt{\tan(x)}} \cdot \sec^2(x)
\end{aligned}$$

If you can simplify this one significantly, you're doing better than I! ■

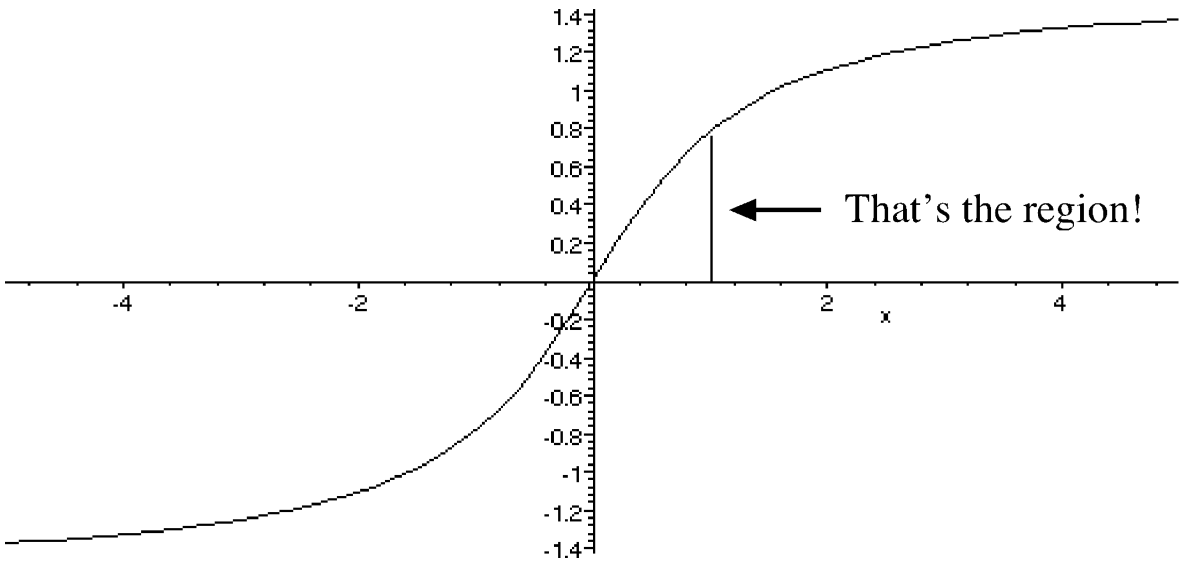
c. Find the area under the parametric curve given by $x = 1 + t^2$ and $y = t(1-t)$ for $0 \le t \le 1$.

Solution. Note that $dx = 2t\,dt$ and that $y = t(1-t) \ge 0$ for $0 \le t \le 1$.

$$\begin{aligned}
\text{Area} &= \int_{t=0}^{t=1} y\,dx = \int_0^1 t(1-t)2t\,dt = 2\int_0^t \left(t^2 - t^3\right)\,dt \\
&= 2\left(\frac{1}{3}t^3 - \frac{1}{4}t^4\right)\Big|_0^1 = 2\left(\frac{1}{3}1^3 - \frac{1}{4}1^4\right) - 2\left(\frac{1}{3}0^3 - \frac{1}{4}0^4\right) = 2\frac{1}{12} = \frac{1}{6}
\end{aligned}$$ ■

d. Sketch the region whose area is computed by the integral $\int_0^1 \arctan(x)\,dx$.

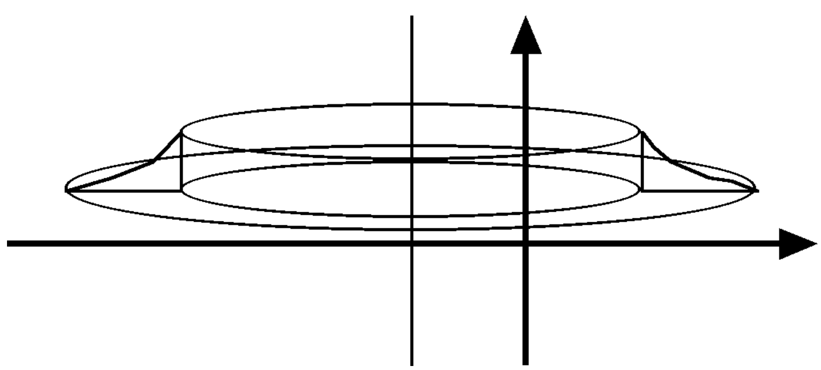
Solution. Note that $\arctan(x) \geq 0$ for $x \geq 0$, $\arctan(0) = 0$, and $\arctan(1) = \frac{\pi}{4}$.



One does need to know what the graph of $\arctan(x)$ looks like; the one above was generated using the MAPLE command `plot(arctan(x),x=-5..5);` (with some additions made in a drawing program). ■

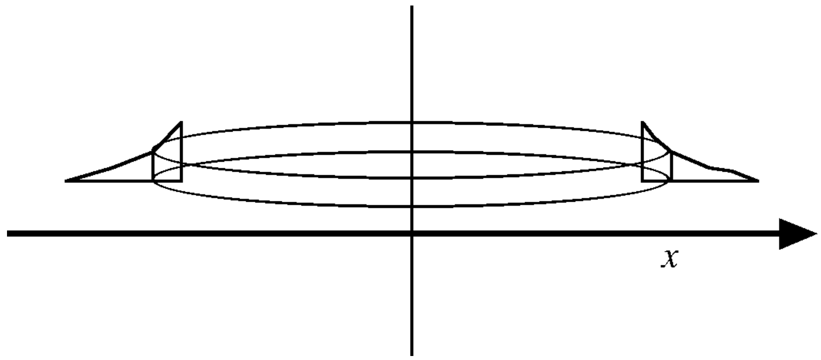
3. Find the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{x}$, $y = \frac{1}{2}$, and $x = 1$ about the line $x = -1$. *[10]*

Solution. Here's a crude sketch of the solid in question:



Note the region that was rotated includes x values from 1 to 2.

We will tackle this problem using shells rather than washers, not that there is much difference in difficulty between the two methods. Since the axis of revolution is a vertical line, the shells are upright and we will need to integrate with respect to the horizontal coordinate axis, namely x. Here is a sketch of the cylindrical shell at x:



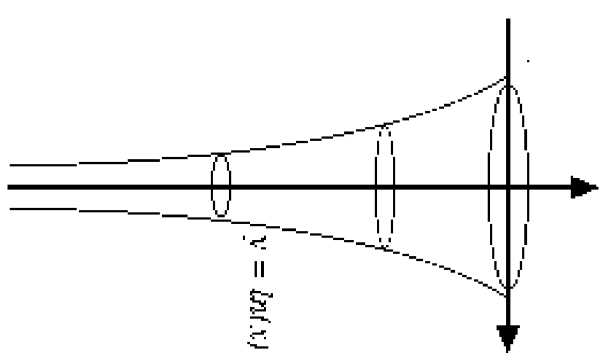
It is not hard to see that this shell has radius $r = x - (-1) = x + 1$ and height $h = \frac{1}{x} - \frac{1}{2}$, and hence area $2\pi r h = 2\pi(x+1)\left(\frac{1}{x} - \frac{1}{2}\right)$.

Thus

$$\begin{aligned}
\text{Volume} &= \int_1^2 2\pi r h \, dx = \int_1^2 2\pi(x+1)\left(\frac{1}{x} - \frac{1}{2}\right) dx = 2\pi \int_1^2 \left(1 - \frac{x}{2} + \frac{1}{x} - \frac{1}{2}\right) dx \\
&= 2\pi \int_1^2 \left(\frac{1}{2} - \frac{x}{2} + \frac{1}{x}\right) dx = 2\pi \left(\frac{x}{2} - \frac{x^2}{4} + \ln(x)\right)\Bigg|_1^2 \\
&= 2\pi\left(\frac{2}{2} - \frac{2^2}{4} + \ln(2)\right) - 2\pi\left(\frac{1}{2} - \frac{1^2}{4} + \ln(1)\right) = 2\pi\left(\ln(2) - \frac{1}{4}\right) \quad \blacksquare
\end{aligned}$$

4. Find the area of the surface obtained by rotating the curve $y = \ln(x)$, $0 < x \leq 1$, about the y-axis. *[10]*

Solution. Here's a crude sketch of the surface:



A slightly nasty feature of this problem is that one must use an improper integral to compute the surface area because $\ln(x)$ has an asymptote at $x = 0$. (Even nastier is the fact that if one does not notice that this requires an improper integral and proceeds blindly using x as the independent variable, one is likely to get the right answer but still lose some marks ...) It should not be too hard to see that the radius of the surface corresponding to the point (x, y) on the curve is just $r = x - 0 = x$. Note that $\frac{dy}{dx} = \frac{d}{dx}\ln(x) = \frac{1}{x}$.

$$A = \int_0^1 2\pi r\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^1 x\sqrt{1+\left(\frac{1}{x}\right)^2}\,dx = 2\pi\int_0^1 x\sqrt{1+\frac{1}{x^2}}\,dx$$

Note that this last *is* an improper integral.

$$= \lim_{t\to 0^+} 2\pi\int_t^1 x\sqrt{1+\frac{1}{x^2}}\,dx = \lim_{t\to 0^+} 2\pi\int_t^1 \sqrt{x^2\left(1+\frac{1}{x^2}\right)}\,dx = \lim_{t\to 0^+} 2\pi\int_t^1 \sqrt{x^2+1}\,dx$$

This is a job for a trig substitution, namely $x = \tan(\theta)$. Then $dx = \sec^2(\theta)\,d\theta$; we'll keep the old limits and substitute back eventually.

$$= \lim_{t\to 0^+} 2\pi\int_{x=t}^{x=1} \sqrt{\tan^2(\theta)+1}\cdot\sec^2(\theta)\,d\theta = \lim_{t\to 0^+} 2\pi\int_{x=t}^{x=1} \sqrt{\sec^2(\theta)}\cdot\sec^2(\theta)\,d\theta$$

$$= \lim_{t\to 0^+} 2\pi\int_{x=t}^{x=1} \sec(\theta)\cdot\sec^2(\theta)\,d\theta = \lim_{t\to 0^+} 2\pi\int_{x=t}^{x=1} \sec^3(\theta)\,d\theta$$

This is an integral we've seen several times over, so we'll just cut to the chase:

$$= \lim_{t\to 0^+} 2\pi\cdot\frac{1}{2}\left(\tan(\theta)\sec(\theta)+\ln|\tan(\theta)+\sec(\theta)|\right)\Big|_{x=t}^{x=1}$$

$$= \lim_{t\to 0^+} \pi\left(x\sqrt{x^2+1}+\ln\left|x+\sqrt{x^2+1}\right|\right)\Big|_t^1$$

$$= \lim_{t\to 0^+} \left[\pi\left(1\sqrt{1^2+1}+\ln\left|1+\sqrt{1^2+1}\right|\right) - \pi\left(t\sqrt{t^2+1}+\ln\left|t+\sqrt{t^2+1}\right|\right)\right]$$

$$= \lim_{t\to 0^+} \left[\pi\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right) - \pi\left(t\sqrt{t^2+1}+\ln\left|t+\sqrt{t^2+1}\right|\right)\right]$$

$$= \pi\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)$$

... because $t\sqrt{t^2+1} \to 0$ as $t \to 0$ and $t+\sqrt{t^2+1} \to 1$, so $\ln\left|t+\sqrt{t^2+1}\right| \to \ln(1) = 0$ as $t \to 0$. ■