Mathematics 110 – Calculus of one variable
TRENT UNIVERSITY, 2003-2004

§A Test #1 Solutions

1. Find $\dfrac{dy}{dx}$ in any *three* of **a-e**. *[12 = 3 × 4 ea.]*

a. $y = x\ln\left(\dfrac{1}{x}\right)$ **b.** $x^2 + 2xy + y^2 - x = 1$ **c.** $y = \sin\left(e^{\sqrt{x}}\right)$

d. $y = \dfrac{2^x}{x+1}$ **e.** $y = \cos(2t)$ where $t = x^3 + 2x$

Solutions.

a. Product rule:

First, the direct approach.

$$\frac{dy}{dx} = \frac{d}{dx}x\ln\left(\frac{1}{x}\right) = \frac{1}{1/x}\cdot\frac{d}{dx}\frac{1}{x} = x\cdot\frac{-1}{x^2} = -\frac{1}{x}$$

Second, an alternative, slightly indirect, approach.

$$y = x\ln\left(\frac{1}{x}\right) = x\ln\left(x^{-1}\right) = -x\ln(x)$$

so

$$\frac{dy}{dx} = \frac{d}{dx}\left(-x\ln(x)\right) = -\frac{d}{dx}\ln(x) = -\frac{1}{x} \qquad \blacksquare$$

b. The direct approach is to use implicit differentiation, plus the product and chain rules along the way:

$$\begin{aligned}
&x^2 + 2xy + y^2 - x = 1 \Longrightarrow \frac{d}{dx}\left(x^2 + 2xy + y^2 - x\right) = \frac{d}{dx}1\\
&\Longrightarrow \frac{d}{dx}x^2 + \frac{d}{dx}2xy + \frac{d}{dx}y^2 - \frac{d}{dx}x = 0 \Longrightarrow 2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} - 1 = 0\\
&\Longrightarrow (2x+2y)\frac{dy}{dx} + (2x + 2y - 1) = 0 \Longrightarrow (2x+2y)\frac{dy}{dx} = 1 - 2x - 2y\\
&\Longrightarrow \frac{dy}{dx} = \frac{1-2x-2y}{2x+2y} = \frac{1}{2x+2y} - 1
\end{aligned}$$

An alternate approach is to solve for y first ...

$$\begin{aligned}
x^2 + 2xy + y^2 - x = 1 &\iff (x+y)^2 - x = 1 \iff (x+y)^2 = 1 + x\\
&\iff x + y = \pm\sqrt{1+x} \iff y = -x \pm \sqrt{1+x}
\end{aligned}$$

… and then take the derivative:

$$\frac{dy}{dx} = \frac{d}{dx}\left(-x \pm \sqrt{1+x}\right) = -1 \pm \frac{1}{2\sqrt{1+x}} \qquad \blacksquare$$

c. Chain rule, twice:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\sin\left(e^{\sqrt{x}}\right) = \cos\left(e^{\sqrt{x}}\right)\cdot\frac{d}{dx}e^{\sqrt{x}} \\ &= \cos\left(e^{\sqrt{x}}\right)e^{\sqrt{x}}\cdot\frac{d}{dx}\sqrt{x} = \cos\left(e^{\sqrt{x}}\right)e^{\sqrt{x}}\frac{1}{2\sqrt{x}} \qquad \blacksquare\end{aligned}$$

d. Quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{2^x}{x+1}\right) = \frac{\frac{d}{dx}2^x\cdot(x+1) - 2^x\cdot\frac{d}{dx}(x+1)}{(x+1)^2} \\ &= \frac{\ln(2)2^x\cdot(x+1) - 2^x\cdot 1}{(x+1)^2} = \frac{2^x\left(\ln(2)(x+1)-1\right)}{(x+1)^2} \qquad \blacksquare\end{aligned}$$

e. Chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{d}{dt}\cos(2t)\cdot\frac{d}{dx}\left(x^3+2x\right) = -\sin(2t)\cdot\frac{d}{dt}2t\cdot\left(3x^2+2\right) \\ &= -2\sin(2t)\cdot\left(3x^2+2\right) = -2\left(3x^2+2\right)\sin\left(2\left(x^3+2x\right)\right) \\ &= -\left(6x^2+4\right)\sin\left(2x^3+4x\right) \qquad \blacksquare\end{aligned}$$

2. Do any *two* of **a-c**. *[10 = 2 × 5 each]*

a. Determine whether $g(x) = \begin{cases} \frac{x-1}{x^2-1} & x \neq 1 \\ \frac{1}{2} & x = 1 \end{cases}$ is continuous at $x = 1$ or not.

Solution. $g(x)$ is continuous at $x = 1$ if and only if $\lim_{x\to 1} g(x)$ exists and equals $g(1)$. Since

$$\lim_{x\to 1} g(x) = \lim_{x\to 1}\frac{x-1}{x^2-1} = \lim_{x\to 1}\frac{x-1}{(x-1)(x+1)} = \lim_{x\to 1}\frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2} = g(1)\,,$$

$g(x)$ is continuous at $x = 1$. $\blacksquare$

b. Use the definition of the derivative to compute $f'(1)$ for $f(x) = \dfrac{1}{x}$.

Solution. Plug in and run …

$$\begin{aligned}f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\frac{1}{1+h}-\frac{1}{1}}{h} = \lim_{h\to 0}\frac{\frac{1-(1+h)}{1+h}}{h} \\ &= \lim_{h\to 0}\frac{\frac{-h}{1+h}}{h} = \lim_{h\to 0}\frac{-h}{h(1+h)} = \lim_{h\to 0}\frac{-1}{1+h} = \frac{-1}{1+0} = -1 \qquad \blacksquare\end{aligned}$$

c. Find the equation of the tangent line to $y = \sqrt{x}$ at $x = 9$.

Solution. Note that at $x = 9$, $y = \sqrt{9} = 3$. The slope m of the tangent line is equal to the derivative of y at $x = 9$.

$$m = \left.\frac{dy}{dx}\right|_{x=9} = \left.\frac{d}{dx}\sqrt{x}\right|_{x=9} = \left.\frac{1}{2\sqrt{x}}\right|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{2\cdot 3} = \frac{1}{6}$$

We want the equation $y = mx + b$ of the line with slope $m = \frac{1}{6}$ passing through the point $(9, 3)$, and it remains to compute the y-intercept, b. We do this by plugging in the slope and the coordinates of the point into the equation of the line and solving for b:

$$3 = \frac{1}{6}\cdot 9 + b \iff 3 = \frac{3}{2} + b \iff b = 3 - \frac{3}{2} = \frac{3}{2}$$

Thus the equation of the tangent line to $y = \sqrt{x}$ at $x = 9$ is $y = \frac{1}{6}x + \frac{3}{2}$. ■

3. Do *one* of **a** or **b**. *[8]*

a. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x\to 2} x^2 = 4$.

Solution. We need to show that for every $\varepsilon > 0$, there is a $\delta > 0$ such that if x is within δ of 2, then x^2 is within ε of 4. As usual, given ε, we try to reverse-engineer the necessary δ:

$$-\varepsilon < x^2 - 4 < \varepsilon \iff -\varepsilon < (x-2)(x+2) < \varepsilon \iff -\frac{\varepsilon}{x+2} < x - 2 < \frac{\varepsilon}{x+2}$$

Unfortunately, δ cannot depend on x, so we need to find a suitable bound for $\frac{\varepsilon}{x+2}$. If we arbitrarily decide to ensure that $\delta \le 1$, then:

$$\begin{aligned} -\delta < x - 2 < \delta &\implies -1 < x - 2 < 1 \implies 1 < x < 3 \implies 3 < x + 2 < 5 \\ &\implies \frac{1}{3} > \frac{1}{x+2} > \frac{1}{5} \implies \frac{\varepsilon}{3} > \frac{\varepsilon}{x+2} > \frac{\varepsilon}{5} \end{aligned}$$

If we now let $\delta = \min\left(1, \frac{\varepsilon}{5}\right)$, this will do the job:

$$\begin{aligned} -\delta < x - 2 < \delta &\implies -\frac{\varepsilon}{5} < x - 2 < \frac{\varepsilon}{5} \qquad \text{because } \delta \le \frac{\varepsilon}{5} \\ &\implies -\frac{\varepsilon}{x+2} < x - 2 < \frac{\varepsilon}{x+2} \\ &\qquad \text{because } -1 \le \delta < x - 2 < \delta \le 1 \text{ implies that } \frac{\varepsilon}{5} < \frac{\varepsilon}{x+2} \\ &\implies -\varepsilon < (x-2)(x+2) < \varepsilon \\ &\implies -\varepsilon < x^2 - 4 < \varepsilon \qquad \ldots \text{ as desired!} \end{aligned}$$

Hence $\lim_{x\to 2} x^2 = 4$. ■

b. Use the $\varepsilon - N$ definition of limits to verify that $\lim_{t\to\infty} \dfrac{1}{t+1} = 0$.

Solution. We need to show that for every $\varepsilon > 0$, there is an $N > 0$ such that if $x > N$, then $\frac{1}{t+1}$ is within ε of 0. Note that as $t \to \infty$, we can assume that $t > -1$, from which it follows that $\frac{1}{t+1} - 0 > 0 > -\varepsilon$. This means we only have to worry about making $\frac{1}{t+1} - 0 < \varepsilon$. As usual, given ε, we try to reverse-engineer the necessary N:

$$\frac{1}{t+1} - 0 < \varepsilon \iff \frac{1}{t+1} < \varepsilon \iff t+1 > \frac{1}{\varepsilon} \iff t > \frac{1}{\varepsilon} - 1$$

Since every step was reversible here, it follows that $N = \dfrac{1}{\varepsilon} - 1$ will do the job. Hence $\lim_{t\to\infty} \dfrac{1}{t+1} = 0$. ∎

4. Find the intercepts, the maximum, minimum, and inflection points, and the vertical and horizontal asymptotes of $f(x) = xe^{-x^2}$ and sketch the graph of $f(x)$ based on this information. *[10]*

Solution.

i. (*Domain*) Since x and e^{-x^2} are defined and continuous for all x, it follows that $f(x) = xe^{-x^2}$ is defined and continuous for all x.

ii. (*Intercepts*) $f(x) = xe^{-x^2} = 0 \iff x = 0$, because $e^{-x^2} > 0$ for all x. Thus $(0,0)$ is the only x-intercept and the only y-intercept of $f(x)$.

iii. (*Local maxima and minima*)

$$f'(x) = \frac{d}{dx}\left(xe^{-x^2}\right) = \frac{d}{dx}x \cdot e^{-x^2} + x \cdot \frac{d}{dx}e^{-x^2} = 1 \cdot e^{-x^2} + x \cdot e^{-x^2} \cdot \frac{d}{dx}\left(-x^2\right)$$
$$= e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x) = \left(1 - 2x^2\right) e^{-x^2}$$

which is also defined for all x.

Since $e^{-x^2} > 0$ for all x,

$$f'(x) = 0 \iff 1 - 2x^2 = 0 \iff x^2 = \frac{1}{2} \iff x = \pm\frac{1}{\sqrt{2}}.$$

We determine which of these give local maxima or minima by considering the intervals of increase and decrease. Note that since $e^{-x^2} > 0$ for all x, $f'(x)$ is positive or negative depending on whether $1 - 2x^2$ is positive or negative.

x	$x < -\frac{1}{\sqrt{2}}$	$x = -\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$	$x = \frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} < x$
$f'(x)$	< 0	0	> 0	0	< 0
$f(x)$	decreasing	local min	increasing	local max	decreasing

Thus $f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}e^{-1/2}$ and $f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-1/2}$ are, respectively, local minimum and local maximum points of $f(x)$.

iv. (*Points of inflection and curvature*)

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}\left[\left(1-2x^2\right)e^{-x^2}\right] = \frac{d}{dx}\left(1-2x^2\right)\cdot e^{-x^2} + \left(1-2x^2\right)\cdot \frac{d}{dx}e^{-x^2}$$
$$= -4xe^{-x^2} + \left(1-2x^2\right)e^{-x^2}\frac{d}{dx}\left(-x^2\right) = -4xe^{-x^2} + \left(1-2x^2\right)e^{-x^2}(-2x)$$
$$= \left(-4x-2x+4x^3\right)e^{-x^2} = \left(4x^3-6x\right)e^{-x^2} = 2x\left(2x^2-3\right)e^{-x^2}$$

which is also defined for all x.

Since $e^{-x^2} > 0$ for all x, $f''(x) = 0$ if $x = 0$ or $x = \pm\sqrt{\frac{3}{2}}$. To sort out the inflection points and intervals of curvature, we check where $f''(x)$ is positive and where it is negative. Again, $e^{-x^2} > 0$ for all x, so $f''(x)$ is positive or negative depending on whether $2x\left(2x^2-3\right)$ is positive or negative

x	$x < -\sqrt{\frac{3}{2}}$	$x = -\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}} < x < 0$	$x = 0$	$0 < x < \sqrt{\frac{3}{2}}$	$x = \sqrt{\frac{3}{2}}$	$x > \sqrt{\frac{3}{2}}$
$f''(x)$	< 0	0	> 0	0	< 0	0	> 0
$f(x)$	conc. down	infl. pt.	conc. up	infl. pt.	conc. down	infl. pt.	conc. up

Thus $f\left(-\sqrt{\frac{3}{2}}\right) = -\sqrt{\frac{3}{2}}e^{-3/2}$, $f(0) = 0$, and $f\left(\sqrt{\frac{3}{2}}\right) = \sqrt{\frac{3}{2}}e^{-3/2}$ are the inflection points of $f(x)$.

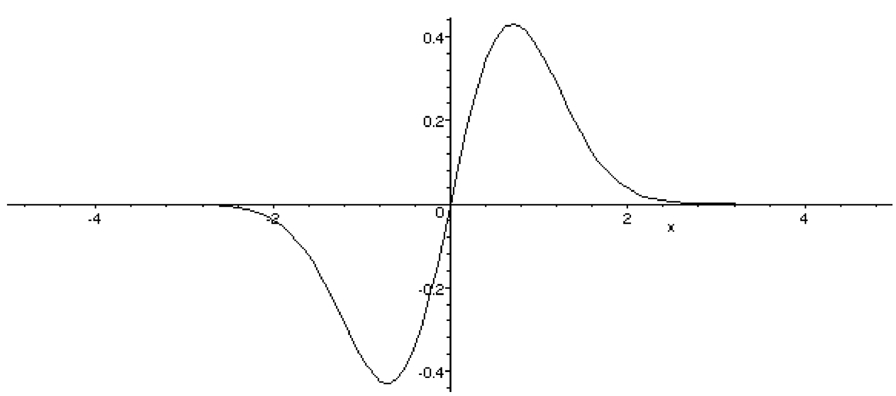
v. (*Vertical asymptotes*) $f(x)$ has no vertical asymptotes because it is defined and continuous for all x.

vi. (*Horizontal asymptotes*) $f(x) = xe^{-x^2} = \frac{x}{e^{x^2}}$ and, since $x \to \pm\infty$ and $e^{x^2} \to infty$ as $x \to \pm\infty$, we can use l'Hôpital's Rule in the relevant limits.

$$\lim_{x\to+\infty} \frac{x}{e^{x^2}} = \lim_{x\to+\infty} \frac{1}{2xe^{x^2}} = 0$$
$$\lim_{x\to-\infty} \frac{x}{e^{x^2}} = \lim_{x\to-\infty} \frac{1}{2xe^{x^2}} = 0$$

Thus $f(x)$ has a horizontal asymptote at $y = 0$ in both directions.

vii. (*The graph!*) Typing `plot(x*exp(-x*x),x=-5..5);` into Maple gives:



Whew! ■