Mathematics 110 – Calculus of one variable
Trent University 2002-2003

QUIZ SOLUTIONS

Quiz #1. (§A) Wednesday, 18 September, 2001. *[10 minutes]*

12:00 Seminar

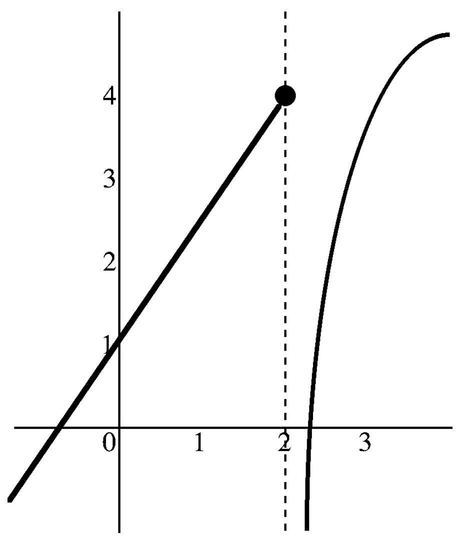
1. Compute $\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$ or show that this limit does not exist. *[5]*

Solution.

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x \to 2}(x+1) = 2 + 1 = 3 \quad \blacksquare$$

2. Sketch the graph of a function $f(x)$ which is defined for all x and for which $\lim_{x \to 0} f(x) = 1$, $\lim_{x \to 2^+} f(x)$ does not exist, and $\lim_{x \to 2^-} f(x) = 4$. *[5]*

Solution.



$\blacksquare$

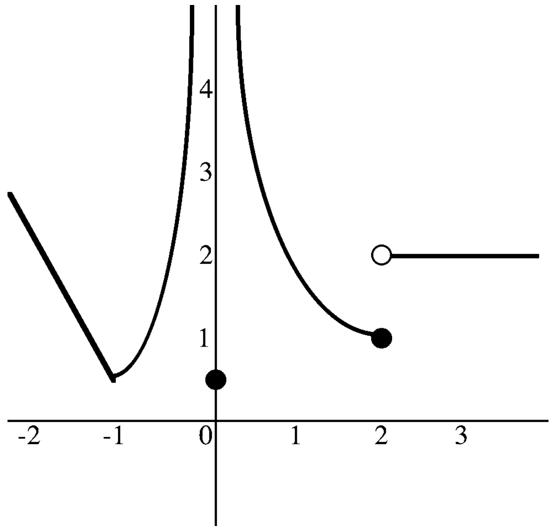
13:00 Seminar

1. Compute $\lim_{x \to 2^-} \frac{x^2 - x + 2}{x - 2}$ or show that this limit does not exist. *[5]*

Solution. Note that $\lim_{x \to 2^-} x^2 - x + 2 = 2^2 - 2 + 2 = 4$ and $\lim_{x \to 2^-} x - 2 = 0$. It follows that $\lim_{x \to 2^-} \frac{x^2 - x + 2}{x - 2}$ fails to exist. (Since when x is a bit less than 2, $x^2 - x + 2$ is about 4 and $x - 2$ is a bit less than 0, it $\lim_{x \to 2^-} \frac{x^2 - x + 2}{x - 2} = -\infty$.) $\blacksquare$

2. Sketch the graph of a function $g(x)$ which is defined for all x, and for which $\lim_{x \to 0} g(x) = \infty$, $\lim_{x \to 2} g(x)$ does not exist, and $g(x)$ does not have an asymptote at $x = 2$. *[5]*

Solution.



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Quiz #2. (§A) Wednesday, 25 September, 2001. *[10 minutes]*

12:00 Seminar

1. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 3}(5x - 7) = 8$. *[10]*

Solution. We need to show that given any $\varepsilon > 0$, one can find a $\delta > 0$ such that if $|x - 3| < \delta$, then $|(5x - 7) - 8| < \varepsilon$. Suppose we are given an $\varepsilon > 0$. Then

$$\begin{aligned} & |(5x-7) - 8| < \varepsilon \\ \iff & |5x - 15| < \varepsilon \\ \iff & 5|x - 3| < \varepsilon \\ \iff & |x - 3| < \frac{\varepsilon}{5}, \end{aligned}$$

so $\delta = \dfrac{\varepsilon}{5}$ will do the job. ■

13:00 Seminar

1. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 2}(3 - 2x) = -1$. *[10]*

Solution. We need to show that given any $\varepsilon > 0$, one can find a $\delta > 0$ such that if $|x - 2| < \delta$, then $|(3 - 2x) - (-1)| < \varepsilon$. Suppose we are given an $\varepsilon > 0$. Then

$$\begin{aligned} & |(3-2x) - (-1)| < \varepsilon \\ \iff & |(3 - 2x) + 1| < \varepsilon \\ \iff & |4 - 2x| < \varepsilon \\ \iff & 2|x - 2| = 2|2 - x| < \varepsilon \\ \iff & |x - 2| < \frac{\varepsilon}{2}, \end{aligned}$$

so $\delta = \frac{\varepsilon}{2}$ will do the job. ■

Quiz #3. Wednesday, 2 October, 2001. *[10 minutes]*

12:00 Seminar

1. For which values of the constant c is the function

$$f(x) = \begin{cases} e^{cx} & x \geq 0 \\ cx + 1 & x < 0 \end{cases}$$

continuous at $x = 0$? Why? *[10]*

Solution. For $f(x)$ to be continuous at $x = 0$ we need to have $f(0)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0^+} f(x)$ all be defined and equal to each other:

$f(0) = e^{c0} = e^0 = 1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} cx + 1 = c0 + 1 = 1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{cx} = e^{c0} = e^0 = 1$

Since all three are defined and equal to 1, no matter what the value of c, $f(x)$ is continuous at $x = 0$. ■

13:00 Seminar

1. For which values of the constant c is the function

$$f(x) = \begin{cases} e^{cx} & x \geq 0 \\ c(x + 1) & x < 0 \end{cases}$$

continuous at $x = 0$? Why? *[10]*

Solution. For $f(x)$ to be continuous at $x = 0$ we need to have $f(0)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0^+} f(x)$ all be defined and equal to each other:

$f(0) = e^{c0} = e^0 = 1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} c(x + 1) = c(0 + 1) = c$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{cx} = e^{c0} = e^0 = 1$

All three are defined for all values of c, but are equal to each other only for $c = 1$. hence $f(x)$ is continuous at $x = 0$ exactly when $c = 1$. ■

Quiz #4. Wednesday, 9 October, 2002. *[12 minutes]*

12:00 Seminar

Suppose

$$f(x) = \begin{cases} x & x < 0 \\ 0 & x = 0 \\ 2x^2 + x & x > 0 \end{cases} .$$

1. Use the definition of the derivative to check whether $f'(0)$ exists and compute it if it does. *[7]*

Solution. By definition, $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$. For this limit to exist, both of $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$ and $\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$ must be defined and be equal:

i. By the definition of $f(x)$, $f(0) = 0$ and $f(x) = x$ when $x < 0$. It follows that:

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(0+h) - 0}{h} = \lim_{h \to 0^-} \frac{h}{h} = \lim_{h \to 0^-} 1 = 1$$

ii. By the definition of $f(x)$, $f(0) = 0$ and $f(x) = 2x^2 + x$ when $x > 0$. It follows that:

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0^+} \frac{\left[2(0+h)^2 + (0+h)\right] - 0}{h} \\ &= \lim_{h \to 0^+} \frac{2h^2 + h}{h} = \lim_{h \to 0^+} (2h + 1) = 1 \end{aligned}$$

Thus $f'(0)$ exists and equals 1. ■

2. Compute $f'(1)$ (any way you like). *[3]*

Solution. By the definition of $f(x)$, $f(x) = 2x^2 + x$ when $x > 0$. Thus, when $x > 0$, $f'(x) = 2 \cdot 2x + 1 = 4x + 1$. Since $1 > 0$ ...), it follows that $f'(1) = 4 \cdot 1 + 1 = 5$. ■

13:00 Seminar

Suppose $g(x) = \frac{1}{x+1}$. Compute $g'(x)$ using

1. the rules for computing derivatives *[3]*, and

Solution. Using the Quotient Rule (and some other bits and pieces ...):

$$\begin{aligned} g'(x) = \frac{d}{dx}\left(\frac{1}{x+1}\right) &= \frac{\left(\frac{d}{dx}1\right) \cdot (x+1) - 1 \cdot \left(\frac{d}{dx}(x+1)\right)}{(x+1)^2} \\ &= \frac{0 \cdot (x+1) - 1 \cdot (1+0)}{(x+1)^2} = \frac{-1}{(x+1)^2} \qquad ■ \end{aligned}$$

2. the definition of the derivative. *[7]*

Solution. Here goes!

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{\frac{1}{(x+h)+1} - \frac{1}{x+1}}{h} = \lim_{h \to 0} \frac{\frac{(x+1)-(x+h+1)}{(x+h)+1)(x+1)}}{h} = \lim_{h \to 0} \frac{-h}{h(x+h)+1)(x+1)} \\ &= \lim_{h \to 0} \frac{-1}{(x+h)+1)(x+1)} = \frac{-1}{(x+1)(x+1)} = \frac{-1}{(x+1)^2} \qquad ■ \end{aligned}$$

Quiz #5. Wednesday, 16 October, 2002. *[10 minutes]*

12:00 Seminar

Compute $\frac{d}{dx}\sqrt[5]{x}$ using

1. the Power Rule *[2]*, and

Solution. $\frac{d}{dx}\sqrt[5]{x} = \frac{d}{dx}x^{1/5} = \frac{1}{5}x^{(1/5)-1} = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}}$ ■

2. the fact that $f(x) = \sqrt[5]{x}$ is the inverse function of $g(x) = x^5$. *[8]*

Solution. Since $f(x) = \sqrt[5]{x}$ is the inverse function of $g(x) = x^5$, $x = g(f(x)) = \left(\sqrt[5]{x}\right)^5$. Differentiating both sides gives:

$$\begin{aligned} 1 = \frac{dx}{dx} &= \frac{d}{dx}\left(\sqrt[5]{x}\right)^5 \\ &= \frac{d}{dx}u^5 \qquad \text{(Where } u = \sqrt[5]{x}.\text{)} \\ &= \left(\frac{d}{du}u^5\right)\cdot\frac{du}{dx} \qquad \text{(Using the Chain Rule.)} \\ &= 5u^4\cdot\frac{du}{dx} \\ &= 5\left(\sqrt[5]{x}\right)^4\cdot\frac{d}{dx}\sqrt[5]{x} \\ &= 5x^{4/5}\cdot\frac{d}{dx}\sqrt[5]{x} \end{aligned}$$

Solving this equation for $\frac{d}{dx}\left(\sqrt[5]{x}\right)$ gives $\frac{d}{dx}\sqrt[5]{x} = \frac{1}{5x^{4/5}}$. ■

13:00 Seminar

1. Compute $\frac{d}{dx}\arccos(x)$ given that $x = \cos(\arccos(x))$ and $\cos^2(x) + \sin^2(x) = 1$. *[10]*

Solution. Differentiating both sides of $x = \cos(\arccos(x))$ gives:

$$\begin{aligned} 1 = \frac{dx}{dx} &= \frac{d}{dx}\cos(\arccos(x)) \\ &= \frac{d}{dx}\cos(u) \qquad \text{(Where } u = \arccos(x).\text{)} \\ &= \left(\frac{d}{du}\cos(u)\right)\cdot\frac{du}{dx} \qquad \text{(Using the Chain Rule.)} \\ &= (-\sin(u))\cdot\frac{du}{dx} \\ &= (-\sin(\arccos(x)))\cdot\frac{d}{dx}\arccos(x) \end{aligned}$$

Solving this equation for $\frac{d}{dx}\arccos(x)$ gives

$$\frac{d}{dx}\arccos(x) = \frac{1}{-\sin(\arccos(x))} = \frac{-1}{\sin(\arccos(x))},$$

which answer can be simplified considerably. Since $\cos^2(x) + \sin^2(x) = 1$, it follows that $\sin(x) = \sqrt{1 - \cos^2(x)}$, so

$$\begin{aligned}\frac{d}{dx}\arccos(x) &= \frac{-1}{\sin\left(\arccos(x)\right)} = \frac{-1}{\sqrt{1-\cos^2\left(\arccos(x)\right)}} \\ &= \frac{-1}{\sqrt{1-\left(\cos\left(\arccos(x)\right)\right)^2}} = \frac{-1}{\sqrt{1-x^2}}\end{aligned}$$

since, once again, $x = \cos\left(\arccos(x)\right)$. ■

Quiz #6. Wednesday, 30 October, 2002. *[10 minutes]*

12:00 Seminar

1. Find the absolute and local maxima and minima of $f(x) = x^3 + 2x^2 - x - 2$ on $[-2, 2]$. *[10]*

Solution. First, note that $f(x)$ is defined and continuous throughout $[-2, 2]$. At the endpoints we get $f(-2) = (-2)^3 + 2 \cdot (-2)^2 - (-2) - 2 = -8 + 8 + 2 - 2 = 0$ and $f(2) = 2^3 + 2 \cdot 2^2 - 2 - 2 = 8 + 8 - 2 - 2 = 12$.

Second, $f'(x) = \frac{d}{dx}\left(x^3 + 2x^2 - x - 2\right) = 3x^2 + 2 \cdot 2x - 1 = 3x^2 + 4x - 1$, which is also defined throughout $[-2, 2]$. To find the critical points we use the quadratic formula:

$$\begin{aligned} f'(x) = 0 &\iff 3x^2 + 4x - 1 = 0 \\ &\iff x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot (-1)}}{2 \cdot 3} \\ &\iff x = \frac{-4 \pm \sqrt{16 + 12)}}{6} \\ &\iff x = \frac{-4 \pm \sqrt{28}}{6} \\ &\iff x = \frac{-4 \pm 2\sqrt{7}}{6} \\ &\iff x = \frac{-2 \pm \sqrt{7}}{3} \end{aligned}$$

The problem here is that $\frac{-2\pm\sqrt{7}}{3}$ is not a terribly nice pair of numbers to play with. One could use a calculator to get results which are close enough for our purposes, or one can use the fact that $2 < \sqrt{7} < 3$ to observe that $-\frac{5}{3} < \frac{-2-\sqrt{7}}{3} < -\frac{4}{3}$ and $0 < \frac{-2+\sqrt{7}}{3} < \frac{1}{3}$. Either way, both $\frac{-2-\sqrt{7}}{3}$ and $\frac{-2+\sqrt{7}}{3}$ are in the interval $[-2, 2]$.

It remains to determine the values of $f(x)$ at the critical points and compare these to each other and to the values at the endpoints. We leave this to the reader; one can use a calculator or approximations to figure out what is going on ... ■

13:00 Seminar

1. Find the absolute and local maxima and minima of $f(x) = x^3 - 3x^2 - x + 3$ on $[-2, 2]$. *[10]*

Solution. First, note that $f(x)$ is defined and continuous throughout $[-2, 2]$. At the endpoints we get $f(-2) = (-2)^3 - 3 \cdot (-2)^2 - (-2) + 3 = -8 - 12 + 2 + 3 = -15$ and $f(2) = 2^3 - 3 \cdot 2^2 - 2 + 3 = 8 - 12 - 2 + 3 = -3$.

Second, $f'(x) = \frac{d}{dx}\left(x^3 - 3x^2 - x + 3\right) = 3x^2 - 3 \cdot 2x - 1 = 3x^2 - 6x - 1$, which is also defined throughout $[-2, 2]$. To find the critical points we use the quadratic formula:

$$\begin{aligned} f'(x) = 0 &\iff 3x^2 - 6x - 1 = 0 \\ &\iff x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot (-1)}}{2 \cdot 3} \\ &\iff x = \frac{6 \pm \sqrt{36 + 12)}}{6} \\ &\iff x = \frac{6 \pm \sqrt{48}}{6} \\ &\iff x = \frac{6 \pm 4\sqrt{3}}{6} \\ &\iff x = \frac{3 \pm 2\sqrt{3}}{3} \end{aligned}$$

The problem here is that $\frac{3 \pm 2\sqrt{3}}{3}$ is not a terribly nice pair of numbers to play with. One could use a calculator to get results which are close enough for our purposes. Using the fact that $1 < \sqrt{3} < 2$ to observe that $-\frac{1}{3} < \frac{3-2\sqrt{3}}{3} < \frac{1}{3}$ and $\frac{5}{3} < \frac{3+2\sqrt{3}}{3} < 3$ does not even tell us whether the second root falls in the interval $[-2, 2]$ ...

It remains to determine the values of $f(x)$ at the critical points and compare these to each other and to the values at the endpoints. We leave this to the reader; one can use a calculator or (better) approximations to figure out what is going on ... ■

Quiz #7. Wednesday, 6 November, 2002. *[15 minutes]*

12:00 Seminar

1. Find the intercepts, critical and inflection points, and horizontal asymptotes of $f(x) = (x - 2)e^x$ and sketch its graph. *[10]*

Solution. Note that $f(x) = (x - 2)e^x$ is defined and continuous everywhere; in particular, it has no vertical asymptotes.

i. Intercepts: For the x-intercept, since $e^x \neq 0$ for all x, $f(x) = 0$ exactly when $x - 2 = 0$, *i.e.* when $x = 2$. For the y-intercept, note that $f(0) = (0 - 2)e^0 = -2 \cdot 1 = -2$.

ii. Critical points: First,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x - 2)e^x = \left(\frac{d}{dx}(x - 2)\right) \cdot e^x + (x - 2)\left(\frac{d}{dx}e^x\right) \\ &= e^x + (x - 2)e^x = (x - 1)e^x , \end{aligned}$$

which is defined and continuous for all x. Since $e^x > 0$ for all x, $f'(x) = 0$ precisely when $x = 1$. Note that because $f'(x) = (x-1)e^x < 0$ for all $x < 1$ and $f'(x) > 0$ for all $x > 1$, $f(x)$ has a local minimum at the critical point.

iii. *Inflection points:* First,

$$f''(x) = \frac{d}{dx}(x-1)e^x = \left(\frac{d}{dx}(x-1)\right) \cdot e^x + (x-1) \cdot \left(\frac{d}{dx}e^x\right)$$
$$= e^x + (x-1)e^x = xe^x ,$$

which is defined and continuous for all x. Since $e^x > 0$ for all x, $f''(x) = 0$ precisely when $x = 0$. Because $f''(x) = xe^x < 0$ for all $x < 0$ and $f''(x) > 0$ for all $x > 0$, $f(x)$ has an inflection point at $x = 0$.

iv. *Horizontal asymptotes:* First,

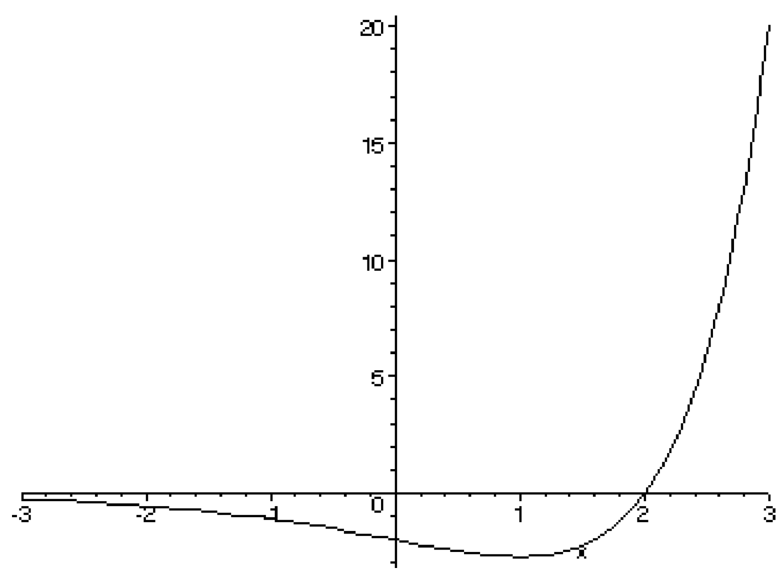
$$\lim_{t \to -\infty} (x-2)e^x = \lim_{t \to \infty} \frac{x-2}{e^{-x}} = \lim_{t \to -\infty} \frac{\frac{d}{dx}(x-2)}{\frac{d}{dx}e^{-x}} = \lim_{t \to -\infty} \frac{1}{e^{-x}} = 0$$

using l'Hôpital's Rule because $\lim_{t \to -\infty} (x-2) = \infty$ and $\lim_{t \to -\infty} e^{-x} = \infty$. It follows that $h(x)$ has a horizontal asymptote of $y = 0$ in the negative x direction. Second,

$$\lim_{t \to \infty} (x-2)e^x = \infty$$

because $\lim_{t \to \infty} (x-2) = \infty$ and $\lim_{t \to \infty} e^x = \infty$. It follows that $h(x)$ has no horizontal asymptote in the positive x direction.

v. *The graph:*



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13:00 Seminar

1. Find the intercepts, critical and inflection points, and horizontal asymptotes of $h(x) = (x+1)e^{-x}$ and sketch its graph. *[10]*

Solution. Note that $h(x) = (x+1)e^{-x}$ is defined and continuous everywhere; in particular, it has no vertical asymptotes.

i. Intercepts: For the x-intercept, since $e^x \neq 0$ for all x, $h(x) = 0$ exactly when $x+1 = 0$, *i.e.* when $x = -1$. For the y-intercept, note that $h(0) = (0-1)e^{-0} = -1 \cdot 1 = -1$.

ii. Critical points: First,

$$\begin{aligned} h'(x) &= \frac{d}{dx}(x+1)e^{-x} = \left(\frac{d}{dx}(x+1)\right) \cdot e^{-x} + (x+1)\left(\frac{d}{dx}e^{-x}\right) \\ &= e^{-x} + (x+1)\left(-e^{-x}\right) = -xe^{-x}\,, \end{aligned}$$

which is defined and continuous for all x. Since $e^x > 0$ for all x, $h'(x) = 0$ precisely when $x = 0$. Note that because $h'(x) = -xe^{-x} > 0$ for all $x < 0$ and $h'(x) < 0$ for all $x > 0$, $h(x)$ has a local maximum at the critical point.

iii. Inflection points: First,

$$\begin{aligned} h''(x) &= \frac{d}{dx}\left(-xe^{-x}\right) = \left(\frac{d}{dx}(-x)\right) \cdot e^{-x} + (-x) \cdot \left(\frac{d}{dx}e^{-x}\right) \\ &= -e^{-x} - x \cdot \left(-e^{-x}\right) = (x-1)e^{-x}\,, \end{aligned}$$

which is defined and continuous for all x. Since $e^x > 0$ for all x, $h''(x) = 0$ precisely when $x = 1$. Because $h''(x) = (x-1)e^{-x} < 0$ for all $x < 1$ and $h''(x) > 0$ for all $x > 1$, $h(x)$ has an inflection point at $x = 1$.

iv. Horizontal asymptotes: First,

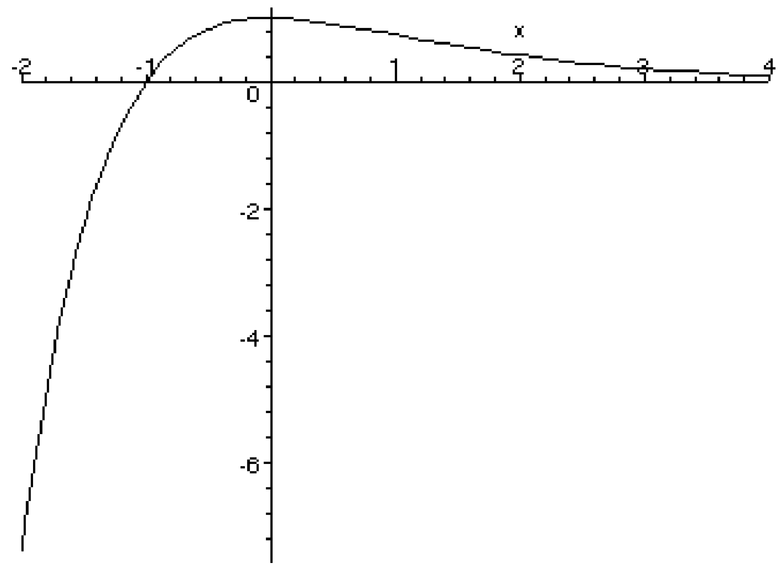
$$\lim_{t \to -\infty} (x+1)e^{-x} = -\infty$$

since $\lim_{t \to -\infty} (x+1) = -\infty$ and $\lim_{t \to -\infty} e^{-x} = \infty$. It follows that $h(x)$ has no horizontal asymptote in the negative x direction. Second,

$$\lim_{t \to \infty} (x+1)e^{-x} = \lim_{t \to \infty} \frac{x+1}{e^x} = \lim_{t \to \infty} \frac{\frac{d}{dx}(x+1)}{\frac{d}{dx}e^x} = \lim_{t \to \infty} \frac{1}{e^x} = 0$$

using l'Hôpital's Rule because $\lim_{t \to \infty} (x+1) = \infty$ and $\lim_{t \to \infty} e^x = \infty$. It follows that $h(x)$ has a horizontal asymptote of $y = 0$ in the positive x direction.

v. The graph:



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Quiz #8. Wednesday, 27 November, 2002. *[15 minutes]*

12:00 Seminar

1. Compute:

$$\int_1^{e^\pi} \frac{1}{x} \sin\left(\ln(x)\right)\, dx \qquad [5]$$

Solution. We'll use the substitution $u = \ln(x)$ and change the limits when we do. Note that $du = \frac{1}{x}dx$ and that when $x = 1$, $u = \ln(1) = 0$, and that when $x = e^\pi$, $u = \ln\left(e^\pi\right) = \pi \ln(e) = \pi \cdot 1 = \pi$. Then

$$\int_1^{e^\pi} \frac{1}{x} \sin\left(\ln(x)\right)\, dx = \int_0^\pi \sin(u)\, du = \cos(u)|_0^\pi = \cos(\pi) - \cos(0) = (-1) - 1 = -2$$

does the job. ■

2. What definite integral does the Right-hand Rule limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right) \cdot \frac{1}{n}$$

correspond to? *[5]*

Solution. The general formula for the Right-hand Rule is:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

Here $\frac{b-a}{n}$ is the width and $f\left(a+i\frac{b-a}{n}\right)$ is the height of the ith rectangle in the Riemann sum using a partition of $[a,b]$ into n equal subintervals and evaluating $f(x)$ at the right endpoint of each subinterval to find the height of the corresponding rectangle.

We compare the right hand side of the formula to the given limit and try to indentify a, b, and f. Comparing the given common width $\frac{1}{n}$ of the rectangles with $\frac{b-a}{n}$ tells us that $b-a=1$; comparing $1+\frac{i}{n}$ with $f\left(a+i\frac{b-a}{n}\right)$, it is a reasonable guess that $a=1$ (so $b=1+1=2$). If so, then $f(x)$ does nothing to x, *i.e.* $f(x)=x$. Hence the desired definite integral could be:

$$\int_1^2 x\,dx$$

This is *not* the only solution. For example, if one had instead guessed that $a=0$ (so $b=1$), we would get that $f(x)=1+x$ and that the integral being sought was

$$\int_0^1 (1+x)\,dx$$

instead. In general, one could let a be any constant; then $b=a+1$ and $f(x)=1-a+x$ do the job. ■

13:00 Seminar

1. Compute:

$$\int_0^{\pi/4} \frac{\tan(x)}{\cos^2(x)}\,dx \qquad [5]$$

Solution. We'll use the fact that $\tan(x)=\frac{\sin(x)}{\cos(x)}$ to rewrite the function and then use the substitution $u=\cos(x)$ and change the limits when we do. Note that $du=-\sin(x)dx$, so $\sin(x)dx=(-1)du$. Also, when $x=0$, $u=\cos(0)=1$, and when $x=\frac{\pi}{4}$, $u=\cos(\pi/4)=\frac{1}{\sqrt{2}}$. Then

$$\begin{aligned}
\int_0^{\pi/4} \frac{\tan(x)}{\cos^2(x)}\,dx &= \int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos^2(x)}\,dx = \int_0^{\pi/4} \frac{\sin(x)}{\cos^3(x)}\,dx \\
&= \int_1^{1/\sqrt{2}} \frac{-1}{u^3}\,du = -\int_1^{1/\sqrt{2}} u^{-3}\,du = -\left.\frac{u^{-2}}{-2}\right|_1^{1/\sqrt{2}} = \left.\frac{1}{2u^2}\right|_1^{1/\sqrt{2}} \\
&= \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)^2} - \frac{1}{2\,(1)^2} = \frac{1}{2\frac{1}{2}} - \frac{1}{2} = \frac{1}{1} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

does the job. ■

2. What definite integral does the Right-hand Rule limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}-1\right)\cdot\frac{1}{n}$$

correspond to? *[5]*

Solution. The general formula for the Right-hand Rule is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

Here $\frac{b-a}{n}$ is the width and $f\left(a + i\frac{b-a}{n}\right)$ is the height of the ith rectangle in the Riemann sum using a partition of $[a,b]$ into n equal subintervals and evaluating $f(x)$ at the right endpoint of each subinterval to find the height of the corresponding rectangle.

We compare the right hand side of the formula to the given limit and try to indentify a, b, and f. Comparing the given common width $\frac{1}{n}$ of the rectangles with $\frac{b-a}{n}$ tells us that $b-a=1$; comparing $\frac{2i}{n}-1$ with $f\left(a + i\frac{b-a}{n}\right)$, one could guess that $a=0$ (so $b=0+1=1$). If so, then $f(x)$ takes x to $2x-1$, *i.e.* $f(x)=2x-1$. Hence the desired definite integral could be:

$$\int_0^1 (2x-1)\,dx$$

This is *not* the only solution. For example, if one had instead guessed that $a=1$ (so $b=2$), we would get that $f(x)=2x-3$ and that the integral being sought was

$$\int_1^2 (2x-3)\,dx$$

instead. In general, one could let a be any constant; then $b=a+1$ and $f(x)=2x-2a-1$ do the job. ■

Quiz #9. Wednesday, 4 December, 2002. *[15 minutes]*

12:00 Seminar

1. Find the area of the region enclosed by $y=-x^2$ and $y=x^2-2x$. *[10]*

Solution. First, we need to find the points of intersection of the two curves. We set $-x^2=x^2-2$, rearrange this to give $2x^2=2$, and observe that the solutions are $x=-1$ and $x=1$.

Second, note that if x is between -1 and 1, $x^2<1$, so $-x^2>-1$ and $x^2-2<1-2=-1$. This tells us that for x between -1 and 1, the curve $y=-x^2$ is above the curve $y=x^2-2$.

Third, the area of the region is thus given by:

$$\begin{aligned}\int_{-1}^1 \left(-x^2-\left(x^2-2\right)\right)\,dx &= \int_{-1}^1 \left(-2x^2+2\right)\,dx = -\frac{2}{3}x^3+2x\Big|_{-1}^1 \\ &= \left(-\frac{2}{3}1^3+2(1)\right)-\left(-\frac{2}{3}(-1)^3+2(-1)\right) \\ &= \frac{4}{3}-\left(\frac{8}{3}\right)=\frac{4}{3}+\frac{8}{3}=\frac{12}{3}=4 \qquad ■\end{aligned}$$

13:00 Seminar

1. Find the area of the region enclosed by $y = (x-2)^2 + 1 = x^2 - 4x + 5$ and $y = x + 1$. *[10]*

Solution. First, we need to find the points of intersection of the two curves. We set $x^2 - 4x + 5 = x + 1$, rearrange this to give $x^2 - 5x + 4 = 0$, and observe that $x^2 - 5x + 4 = (x-4)(x-1)$. (Worst coming to worst, one could get that by using the quadratic formula.) It follows that the solutions are $x = 1$ and $x = 4$.

Second, note that if x is between 1 and 4, $x^2 - 5x + 4 = \left(x^2 - 4x + 5\right) - (x+1) < 0$, because the parabola $y = x^2 - 5x + 4$ opens upwards and its tip must be between its zeros. This tells us that for x between 1 and 4, the line $y = x + 1$ is above the curve $y = x^2 - 4x + 5$.

Third, the area of the region is thus given by:

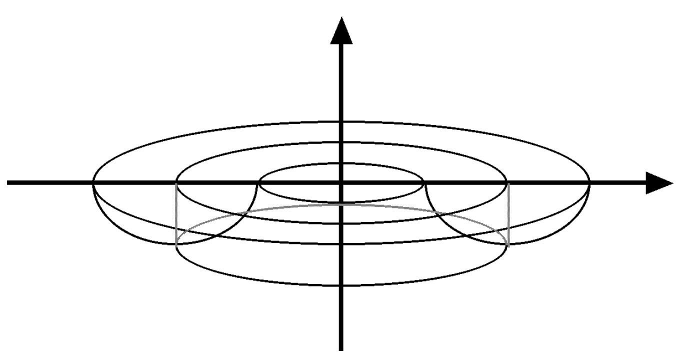
$$\begin{aligned}\int_1^4 \left((x+1) - \left(x^2 - 4x + 5\right)\right)\,dx &= \int_1^4 \left(-x^2 + 5x - 4\right)\,dx = -\frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x\Big|_1^4 \\ &= \left(-\frac{1}{3}4^3 + \frac{5}{2}4^2 - 4\cdot 4\right) - \left(-\frac{1}{3}1^3 + \frac{5}{2}1^2 - 4\cdot 1\right) \\ &= \frac{44}{3} - \frac{11}{6} = \frac{77}{6} \qquad \blacksquare\end{aligned}$$

Quiz #10. Wednesday, 8 January, 2003. *[25 minutes]*

12:00 Seminar

1. Sketch the solid obtained by rotating the region bounded by $y = 0$ and $y = \cos(x)$ for $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ about the y-axis and find its volume. *[10]*

Solution. Observe that $\cos(x) \le 0$ for $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$. The solid in question looks like this:



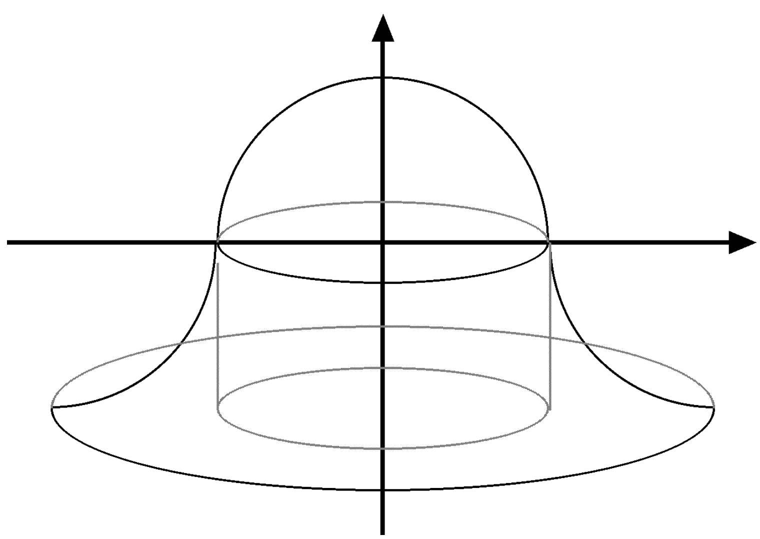
We will find the volume of this solid using the method of cylindrical shells. Since we rotated about a vertical line, we will use x as the variable. Note that the cylinder whose edge passes through x has height $h = 0 - \cos(x) = -\cos(x)$ and radius $r = x - 0 = x$. The volume of the solid is:

$$\begin{aligned}
V &= \int_{\pi/2}^{3\pi/2} 2\pi r h \, dx \\
&= \int_{\pi/2}^{3\pi/2} 2\pi x \left(-\cos(x)\right) \, dx \\
&= -2\pi \int_{\pi/2}^{3\pi/2} x\cos(x)\, dx \\
&\quad \text{We use integration by parts with } u = x \text{ and } dv = \cos(x)\, dx, \\
&\quad \text{so } du = dx \text{ and } v = \sin(x). \\
&= -2\pi \left[x\sin(x)\Big|_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin(x)\, dx \right] \\
&= -2\pi \left[\left(\frac{3\pi}{2}\sin\left(\frac{3\pi}{2}\right) - \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) \right) - \left(-\cos(x)\Big|_{\pi/2}^{3\pi/2}\right) \right] \\
&= -2\pi \left[\left(\frac{3\pi}{2}(-1) - \frac{\pi}{2}1 \right) + \left(\cos\left(\frac{3\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right) \right) \right] \\
&= -2\pi \left[-\frac{4}{2}\pi - (0 - 0) \right] \\
&= 4\pi^2 \qquad \blacksquare
\end{aligned}$$

13:00 Seminar

1. Sketch the solid obtained by rotating the region bounded by $y = -1$ and $y = \cos(x)$ for $0 \le x \le \pi$ about the y-axis and find its volume. *[10]*

Solution. Observe that $\cos(x) \ge -1$ for $0 \le x \le \pi$. The solid in question looks like this:



We will find the volume of this solid using the method of cylindrical shells. Since we rotated about a vertical line, we will use x as the variable. Note that the cylinder whose edge passes through x has height $h = \cos(x) - (-1) = \cos(x) + 1$ and radius $r = x - 0 = x$. The volume of the solid is:

$$\begin{aligned}
V &= \int_0^\pi 2\pi r h \, dx \\
&= \int_0^\pi 2\pi x \left(\cos(x) + 1\right) \, dx \\
&= 2\pi \int_0^\pi x \left(\cos(x) + 1\right) \, dx
\end{aligned}$$

We use integration by parts with $u = x$ and $dv = (\cos(x) + 1)\, dx$, so $du = dx$ and $v = \sin(x) + x$.

$$\begin{aligned}
&= 2\pi \left[x\left(\sin(x) + x\right)\Big|_0^\pi - \int_0^\pi \left(\sin(x) + x\right) \, dx \right] \\
&= 2\pi \left[\left(\pi\left(\sin(\pi) + \pi\right) - 0\left(\sin(0) + 0\right)\right) - \left(-\cos(x) + \frac{1}{2}x^2 \Big|_0^\pi \right) \right] \\
&= 2\pi \left[\left(\pi(0 + \pi) - 0\right) - \left(\left(-\cos(\pi) + \frac{1}{2}\pi^2 \right) - \left(-\cos(0) + \frac{1}{2}0^2 \right) \right) \right] \\
&= 2\pi \left[\pi^2 - \left(\left(-(-1) + \frac{1}{2}\pi^2 \right) - (-1 - 0) \right) \right] \\
&= 2\pi \left[\pi^2 - \left(2 + \frac{1}{2}\pi^2 \right) \right] \\
&= 2\pi \left[\frac{1}{2}\pi^2 - 2 \right] \\
&= \pi^3 - 4\pi \qquad \blacksquare
\end{aligned}$$

Quiz #11. Wednesday, 15 January, 2002. *[20 minutes]*

12:00 Seminar

1. Compute $\displaystyle\int \frac{1}{1 - x^2} \, dx$.

Solution. We'll use the trigonometric substitution $x = \sin(t)$; note that then $dx = \cos(t)\, dt$ and $\cos(t) = \sqrt{1 - x^2}$.

$$\begin{aligned}
\int \frac{1}{1 - x^2} \, dx &= \int \frac{1}{1 - \sin^2(t)} \cos(t) \, dt = \int \frac{1}{\cos^2(t)} \cos(t) \, dt \\
&= \int \frac{1}{\cos(t)} \, dt = \int \sec(t) \, dt = \ln\left(\sec(t) + \tan(t)\right) + C \\
&= \ln\left(\frac{1}{\cos(t)} + \frac{\sin(t)}{\cos(t)} \right) + C = \ln\left(\frac{1}{\sqrt{1 - x^2}} + \frac{x}{\sqrt{1 - x^2}} \right) + C \qquad \blacksquare
\end{aligned}$$

13:00 Seminar

1. Compute $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$.

Solution. We'll use the trigonometric substitution $x = \sin(t)$; note that then $dx = \cos(t)\,dt$, $t = \arcsin(x)$, and $\cos(t) = \sqrt{1-x^2}$.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^2(t)}{\sqrt{1-\sin^2(t)}}\cos(t)\,dt = \int \frac{\sin^2(t)}{\cos(t)}\cos(t)\,dt \\
&= \int \sin^2(t)\,dt = \int \frac{1}{2}\left(1-\cos(2t)\right)\,dt = \frac{1}{2}\left(t+\frac{1}{2}\sin(2t)\right)+C \\
&= \frac{1}{2}t+\frac{1}{4}2\sin(t)\cos(t)+C = \frac{1}{2}\arcsin(x)+\frac{1}{2}x\sqrt{1-x^2}+C \qquad \blacksquare
\end{aligned}$$

Quiz #12. Wednesday, 22 January, 2002. *[20 minutes]*

12:00 Seminar

1. Compute $\int \frac{3x^2+4x+2}{x^3+2x^2+2x}\,dx$.

Solution. This can be done by partial fractions, but since $\frac{d}{dx}\left(x^3+2x^2+2x\right) = 3x^2 + 4x+2$, there is a quicker alternative, namely the substitution $u = x^3+2x^2+2x$:

$$\int \frac{3x^2+4x+2}{x^3+2x^2+2x}\,dx = \int \frac{1}{u}\,du = \ln(u)+C = \ln\left(x^3+2x^2+2x\right)+C$$

Not many people spotted this shortcut ... $\blacksquare$

13:00 Seminar

1. Compute $\int \frac{2x+1}{x^3+2x^2+x}\,dx$.

Solution. Since we are trying to integrate a rational function and there are no readily apparent shortcuts, we use partial fractions.

First, note that the degree of the numerator is already less than the degree of the denominator.

Second, we factor the numerator as far as possible:

$$x^3+2x^2+x = x\left(x^2+2x+1\right) = x\left(x+1\right)^2$$

Third, it follows that

$$\frac{2x+1}{x^3+2x^2+x} = \frac{2x+1}{x\left(x+1\right)^2} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

for some constants A, B, and C. Putting the right-hand side over a common denominator of $x\left(x+1\right)^2$ and comparing numerators, we see that we must have:

$$\begin{aligned}
2x+1 &= A(x+1)^2+Bx(x+1)+Cx \\
&= A\left(x^2+2x+1\right)+B\left(x^2+x\right)+Cx \\
&= (A+B)x^2+(2A+B+C)x+A
\end{aligned}$$

Hence $A + B = 0$, $2A + B + C = 2$, and $A = 1$, from which it follows pretty quickly that $B = -1$ and $C = 1$.

Fourth, we compute the integral:

$$\begin{aligned}
\int \frac{2x+1}{x^3+2x^2+x}\,dx &= \int \left(\frac{1}{x} + \frac{-1}{x+1} + \frac{1}{(x+1)^2}\right) dx \\
&= \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx \\
&= \ln(x) - \ln x + 1 + \frac{-1}{x+1} + K \\
&= \ln\left(\frac{x}{x+1}\right) - \frac{1}{x+1} + K
\end{aligned}$$

We're using K for the generic constant because C has already been used ... ■

Quiz #13. Wednesday, 29 January, 2002. *[15 minutes]*

12:00 Seminar

1. Compute $\int_{-\infty}^{\infty} e^{-|x|}\,dx$ or show that it does not converge. *[10]*

Solution. This is obviously an improper integral since there is an infinity in each limit of integration.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-|x|}\,dx &= \int_{-\infty}^{0} e^{-|x|}\,dx + \int_{0}^{\infty} e^{-|x|}\,dx \\
&= \int_{-\infty}^{0} e^{-(-x)}\,dx + \int_{0}^{\infty} e^{-x}\,dx \\
&\qquad \ldots \text{ since } |a| = -a \text{ when } a \le 0 \text{ and } |a| = a \text{ when } a \ge 0. \\
&= \lim_{t\to-\infty} \int_{t}^{0} e^{x}\,dx + \lim_{t\to\infty} \int_{0}^{t} e^{-x}\,dx \\
&= \lim_{t\to-\infty} e^{x}\Big|_{t}^{0} + \lim_{t\to\infty} -e^{-x}\Big|_{0}^{t} \\
&= \lim_{t\to-\infty} \left(e^{0} - e^{t}\right) + \lim_{t\to\infty} \left(\left(-e^{-t}\right) - \left(-e^{-0}\right)\right) \\
&= \lim_{t\to-\infty} \left(1 - e^{t}\right) + \lim_{t\to\infty} \left(1 - e^{-t}\right) \\
&= 1 + 1 = 2
\end{aligned}$$

Note that $\lim_{t\to-\infty} e^{t} = \lim_{t\to\infty} e^{-t} = 0$. ■

13:00 Seminar

1. Compute $\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x}}\,dx$ or show that it does not converge. *[10]*

Solution. This is an improper integral since $f(x) = \frac{x+1}{\sqrt[3]{x}}$ has an asymptote at $x = 0$, which is in the interval over which the integral is taken. We will do a little bit of algebra first to simplify our task.

$$\begin{aligned}
\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x}}\,dx &= \int_{-1}^{1} \left(\frac{x}{\sqrt[3]{x}} + \frac{1}{\sqrt[3]{x}}\right)\,dx \\
&= \int_{-1}^{1} \left(\frac{x}{x^{1/3}} + \frac{1}{x^{1/3}}\right)\,dx \\
&= \int_{-1}^{1} \left(x^{2/3} + x^{-1/3}\right)\,dx \\
&= \int_{-1}^{1} x^{2/3}\,dx + \int_{-1}^{1} x^{-1/3}\,dx \\
&\quad \text{Note that the left integral is not improper.} \\
&= \frac{3}{5}x^{5/3}\Big|_{-1}^{1} + \int_{-1}^{0} x^{-1/3}\,dx + \int_{0}^{1} x^{-1/3}\,dx \\
&= \left(\frac{3}{5}1^{5/3} - \frac{3}{5}(-1)^{5/3}\right) + \lim_{t\to 0^-}\int_{-1}^{t} x^{-1/3}\,dx + \lim_{t\to 0^+}\int_{t}^{1} x^{-1/3}\,dx \\
&= \left(\frac{3}{5} - \left(-\frac{3}{5}\right)\right) + \lim_{t\to 0^-} \frac{3}{2}x^{2/3}\Big|_{-1}^{t} + \lim_{t\to 0^+} \frac{3}{2}x^{2/3}\Big|_{t}^{1} \\
&= \left(\frac{3}{5} + \frac{3}{5}\right) + \lim_{t\to 0^-}\left(\frac{3}{2}t^{2/3} - \frac{3}{2}(-1)^{2/3}\right) + \lim_{t\to 0^+}\left(\frac{3}{2}1^{2/3} - \frac{3}{2}t^{2/3}\right) \\
&= \frac{6}{5} + \lim_{t\to 0^-}\left(\frac{3}{2}t^{2/3} - \frac{3}{2}\right) + \lim_{t\to 0^+}\left(\frac{3}{2} - \frac{3}{2}t^{2/3}\right) \\
&= \frac{6}{5} + \left(0 - \frac{3}{2}\right) + \left(\frac{3}{2} - 0\right) \\
&= \frac{6}{5} - \frac{3}{2} + \frac{3}{2} \\
&= \frac{6}{5}
\end{aligned}$$

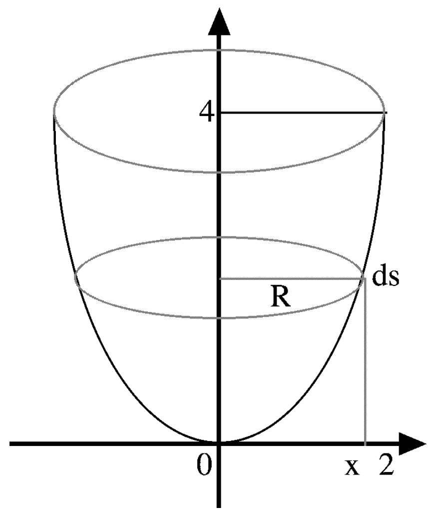
Note that $\lim_{t\to 0^-} t^{2/3} = \lim_{t\to 0^+} t^{2/3} = 0$. ■

Quiz #14. Wednesday, 5 February, 2002. *[20 minutes]*

12:00 Seminar

1. Sketch the solid obtained by rotating the region bounded by $x = 0$, $y = 4$ and $y = x^2$ for $0 \le x \le 2$ about the y-axis. *[2]*

Solution.



■

2. Compute the surface area of this solid. *[8]*

Solution. The surface of the solid has two parts: the disk at the top and the parabolic surface below it. The disk has radius 2 and hence has area $\pi 2^2 = 4\pi$. We compute the area of the parabolic surface using the formula for the area of a surface of revolution. Note that the radius R for the infinitesimal arc length at x is simply $R = x - 0 = x$. Then the area of the parabolic surface is given by:

$$\int_0^2 2\pi R\, ds = \int_0^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \pi \int_0^2 2x \sqrt{1 + \left(\frac{d}{dx}x^2\right)^2}\, dx$$

$$= \pi \int_0^2 2x\sqrt{1 + (2x)^2}\, dx = \pi \int_0^2 2x\sqrt{1 + 4x^2}\, dx$$

Using the substitution $u = 1 + 4x^2$ we get $du = 8xdx$, so $\frac{1}{4}du = 2xdx$. When $x = 0$, $u = 1$, and when $x = 2$, $u = 17$.

$$= \pi \int_1^{17} \sqrt{u}\, \frac{1}{4} du = \frac{\pi}{4}\, \frac{1}{2\sqrt{u}}\Bigg|_1^{17}$$

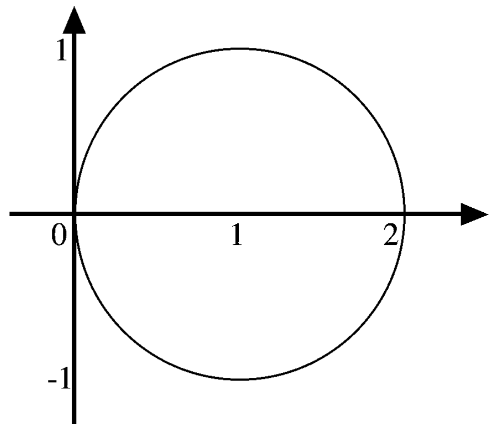
$$= \frac{\pi}{4}\left(\frac{1}{2\sqrt{17}} - \frac{1}{2\sqrt{1}}\right) = \frac{\pi}{8}\left(\frac{1}{\sqrt{17}} - 1\right)$$

Thus the total surface area of the solid is $4\pi + \frac{\pi}{8}\left(\frac{1}{\sqrt{17}} - 1\right) = \frac{\pi}{8}\left(31 + \sqrt{17}\right)$. ■

13:00 Seminar

1. Sketch the curve given by the parametric equations $x = 1 + \cos(t)$ and $y = \sin(t)$, where $0 \leq t \leq 2\pi$. *[3]*

Solution.



■

2. Compute the arc-length of this curve using a suitable integral. *[7]*

Solution. We'll use the parametric version of the arc-length formula:

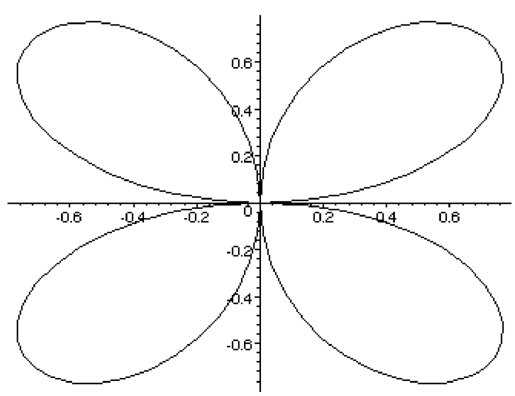
$$\begin{aligned}\int_0^{2\pi} ds &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\left(\frac{d}{dt}\left((1+\cos(t)\right)\right)^2 + \left(\frac{d}{dt}\sin(t)\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(0-\sin(t))^2 + (\cos(t))^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2(t)+\cos^2(t)}\, dt \\ &= \int_0^{2\pi} \sqrt{1}\, dt = \int_0^{2\pi} 1\, dt = t\big|_0^{2\pi} = 2\pi - 0 = 2\pi \qquad \blacksquare\end{aligned}$$

Quiz #15. Wednesday, 26 February, 2002. *[20 minutes]*

12:00 Seminar

1. Graph the polar curve $r = \sin(2\theta)$, $0 \leq \theta \leq 2\pi$. *[4]*

Solution.



■

2. Find the area of the region enclosed by this curve. *[6]*

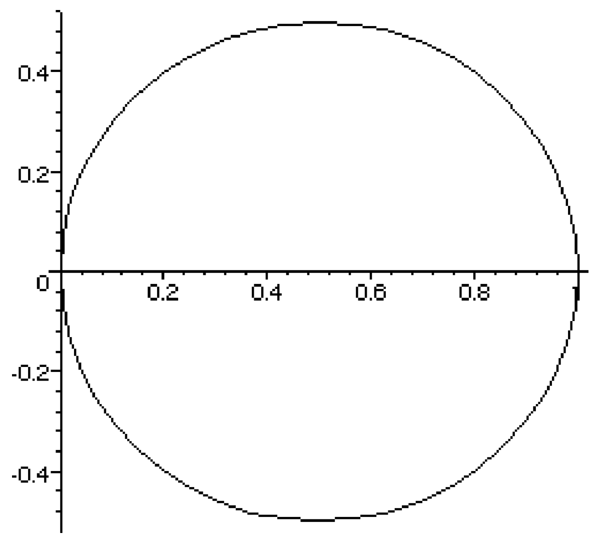
Solution. The area can be computed as follows:

$$\begin{aligned}\int_0^{2\pi} \frac{1}{2} r^2\, d\theta &= \int_0^{2\pi} \frac{1}{2}\sin^2(2\theta)\, d\theta = \frac{1}{2}\int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos(2\cdot 2\theta)\right) d\theta \\ &= \frac{1}{4}\int_0^{2\pi} (1-\cos(4\theta))\, d\theta = \frac{1}{4}\left(\theta - \frac{1}{4}\sin(4\theta)\right)\Big|_0^{2\pi} \\ &= \frac{1}{4}\left(2\pi - \frac{1}{4}\sin(4\cdot 2\pi)\right) - \frac{1}{4}\left(0 - \frac{1}{4}\sin(0\cdot 2\pi)\right) \\ &= \frac{1}{4}\left(2\pi - \frac{1}{4}\sin(8\pi)\right) - \frac{1}{4}\left(0 - \frac{1}{4}\sin(0)\right) \\ &= \frac{1}{4}(2\pi - 0) - (0-0) = \frac{pi}{2} \qquad \blacksquare\end{aligned}$$

13:00 Seminar

1. Graph the polar curve $r = \cos(\theta)$, $0 \le \theta \le 2\pi$. *[4]*

Solution.



$\blacksquare$

2. Find the arc-length of this curve. *[6]*

Solution. Note that to find the arc-length of the curve we only need to trace it once, so we only need $0 \le \theta \le \pi$. The arc-length can then be computed as follows:

$$\begin{aligned}\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta &= \int_0^{\pi} \sqrt{\cos^2(\theta) + (-\sin(\theta))^2}\, d\theta \\ &= \int_0^{\pi} \sqrt{\cos^2(\theta) + \sin^2(\theta)}\, d\theta = \int_0^{\pi} \sqrt{1}\, d\theta \\ &= \int_0^{\pi} 1\, d\theta = \theta\big|_0^{\pi} = \pi - 0 = \pi \qquad \blacksquare\end{aligned}$$

Quiz #16. Wednesday, 5 March, 2003. *[15 minutes]*

12:00 Seminar

Let $a_k = \dfrac{1}{(k+1)(k+2)}$ and $s_n = \displaystyle\sum_{k=0}^{n} a_k$.

1. Find a formula for s_n in terms of n. *[5]*

Solution. Note that $\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}$. (Partial fractions!) Hence:

$$\begin{aligned} s_n &= \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n} \left[\frac{1}{k+1} - \frac{1}{k+2}\right] \\ &= \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \cdots + \left[\frac{1}{n} - \frac{1}{n+1}\right] + \left[\frac{1}{n+1} - \frac{1}{n+2}\right] \\ &= \frac{1}{1} - \frac{1}{n+2} = 1 - \frac{1}{n+2} \qquad \blacksquare \end{aligned}$$

2. Does $\displaystyle\sum_{k=0}^{\infty} a_k$ converge? If so, what does it converge to? *[5]*

Solution. It converges to 1:

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+2}\right) = 1 - 0 = 1 \qquad \blacksquare$$

13:00 Seminar

Let $a_k = \ln\left(\dfrac{k}{k+1}\right)$ and $s_n = \displaystyle\sum_{k=1}^{n} a_k$.

1. Find a formula for s_n in terms of n. *[5]*

Solution. The key here is that $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$.

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \ln\left(\frac{k}{k+1}\right) = \sum_{k=1}^{n} [\ln(k) - \ln(k+1)] \\ &= [\ln(1) - \ln(2)] + [\ln(2) - \ln(3)] + [\ln(3) - \ln(4)] + \cdots \\ &\qquad + [\ln(n-1) - \ln(n)] + [\ln(n) - \ln(n+1)] \\ &= \ln(1) - \ln(n+1) = 0 - \ln(n+1) = -\ln(n+1) \qquad \blacksquare \end{aligned}$$

2. Does $\displaystyle\sum_{k=0}^{\infty} a_k$ converge? If so, what does it converge to? *[5]*

Solution. It does not converge. First, recall that $\ln(x) \to \infty$ as $x \to \infty$. Now:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} [-\ln(n+1)] = -\lim_{n\to\infty} \ln(n+1) = -\infty \qquad \blacksquare$$

Quiz #17. Wednesday, 12 March, 2003. *[15 minutes]*

12:00 Seminar

Determine whether each of the following series converges or diverges:

$$1. \quad \sum_{n=0}^{\infty} e^{-n} \quad [5] \qquad 2. \quad \sum_{n=1}^{\infty} \frac{1}{\arctan(n)} \quad [5]$$

Solution for 1. $\displaystyle\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \frac{1}{e^n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ converges because this is a geometric series with $|r| = \left|\frac{1}{e}\right| < 1$. ■

Solution for 2. Since

$$\lim_{n\to\infty} \frac{1}{\arctan(n)} = \frac{1}{\lim_{n\to\infty} \arctan(n)} = \frac{1}{\pi/2} = \frac{2}{\pi} \neq 0\,,$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\arctan(n)}$ diverges by the Divergence Test. ■

13:00 Seminar

Determine whether each of the following series converges or diverges:

$$1. \quad \sum_{n=0}^{\infty} \frac{1}{n+1} \quad [5] \qquad 2. \quad \sum_{n=1}^{\infty} 2^{1/n^2} \quad [5]$$

Solution for 1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by the (general) p-test since the degree of the denominator is 1, the degree of the numerator is 0, and their difference, $1 - 0 = 1$, is not greater than 1. (The divergence of this series can also be verified pretty quickly using the Integral Test.) ■

Solution for 2. Since

$$\lim_{n\to\infty} 2^{1/n^2} = 2^{\lim_{n\to\infty} 1/n^2} = 2^0 = 1 \neq 0\,,$$

$\displaystyle\sum_{n=1}^{\infty} 2^{1/n^2}$ diverges by the Divergence Test. ■

Quiz #18. Wednesday, 17 March, 2003. *[15 minutes]*

12:00 Seminar

Determine whether each of the following series converges absolutely, converges conditionally, or diverges:

$$1. \quad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n^2+2} \quad [5] \qquad 2. \quad \sum_{n=1}^{\infty} \frac{n!(-1)^n}{n^n} \quad [5]$$

Solution for 1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n^2+2}$ diverges by the Divergence Test: since

$$\lim_{n\to\infty} \left| \frac{(-1)^n 2^n}{n^2+2} \right| = \lim_{n\to\infty} \frac{2^n}{n^2+2} = \lim_{x\to\infty} \frac{2^x}{x^2+2} = \lim_{x\to\infty} \frac{\ln(2)2^x}{2x+0} = \lim_{x\to\infty} \frac{(\ln(2))^2\, 2^x}{2} = \infty$$

(using l'Hôpital's Rule twice), it follows that $\displaystyle\lim_{n\to\infty} \frac{(-1)^n 2^n}{n^2+2}$ does not exist. ■

Solution for 2. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(-1)^n}{n^n}$ converges absolutely since $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by the Comparison Test:

$$\frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdot \ldots \cdot 3 \cdot 2 \cdot 1}{n \cdot n \cdot n \cdot \ldots \cdot n \cdot n \cdot n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n} \leq 1 \cdot \frac{2}{n} \cdot \frac{1}{n} = \frac{2}{n^2}$$

and $\displaystyle\sum_{n=1} \frac{2}{n^2} = 2 \sum_{n=1} \frac{1}{n^2}$ converges by the p-test because $2 - 0 > 1$. ■

13:00 Seminar

Determine whether each of the following series converges absolutely, converges conditionally, or diverges:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(2n^2+3n+4\right)}{3n^2+4n+5}$ *[5]*
2. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2}$ *[5]*

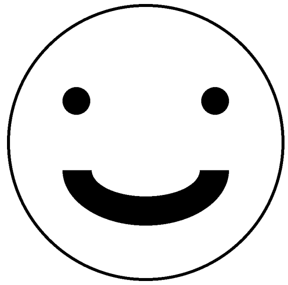
Solution for 1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(2n^2+3n+4\right)}{3n^2+4n+5}$ diverges by the Divergence Test: since

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{(-1)^n \left(2n^2+3n+4\right)}{3n^2+4n+5} \right| &= \lim_{n\to\infty} \frac{2n^2+3n+4}{3n^2+4n+5} = \lim_{n\to\infty} \frac{2n^2+3n+4}{3n^2+4n+5} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{2+\frac{3}{n}+\frac{4}{n^2}}{3+\frac{4}{n}+\frac{5}{n^2}} = \frac{2+0+0}{3+0+0} = \frac{2}{3} \end{aligned}$$

it follows that $\displaystyle\lim_{n\to\infty} \frac{(-1)^n \left(2n^2+3n+4\right)}{3n^2+4n+5} \neq 0$. ■

Solution for 2. Note that $\cos(\pi) = -1$, $\cos(2\pi) = 1 = (-1)^2$, $\cos(3\pi) = -1 = (-1)^3$, $\cos(4\pi) = 1 = (-1)^4$, and so on. It follows that the given series is the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. This converges absolutely because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test since $2-0 > 1$. ■

Bonus Quiz. Friday, 19 March, 2003. *[15 minutes]*



1. A smiley face is drawn on the surface of a balloon which is being inflated at a rate of 10 cm^3/s. At the instant that the radius of the balloon is 10 cm the eyes are 10 cm apart, as measured *inside* the balloon. How is the distance between them changing at this moment? *[10]*

Solution. ■

Quiz #19. Wednesday, 24 March, 2003. *[20 minutes]*

12:00 Seminar

Consider the power series $\sum_{n=0}^{\infty} \frac{2^n x^{2n}}{n!}$.

1. For which values of x does this series converge? *[6]*

Solution. ■

2. This series is equal to a (reasonably nice) function. What is it? Why? *[4]*

Solution. ■

13:00 Seminar

Consider the power series $\sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n+1}$.

1. For which values of x does this series converge? *[6]*

Solution. ■

2. This series is equal to a (reasonably nice) function. What is it? Why? *[4]*

Solution. ■

Quiz #20. Wednesday, 2 April, 2003. *[20 minutes]*

12:00 Seminar

Let $f(x) = \sin(\pi - 2x)$.

1. Find the Taylor series at $a = 0$ of $f(x)$. *[6]*

Solution. ■

2. Find the radius and interval of convergence of this Taylor series. *[4]*

Solution. ■

13:00 Seminar

Let $f(x) = \ln(2 + x)$.

1. Find the Taylor series at $a = 0$ of $f(x)$. *[6]*

Solution. ■

2. Find the radius and interval of convergence of this Taylor series. *[4]*

Solution. ■