Mathematics 110 – Calculus of one variable

Trent University 2001-2002

SOLUTIONS TO ASSIGNMENT #7

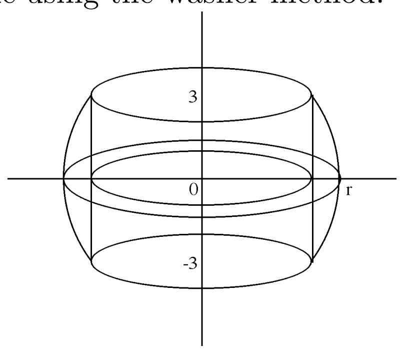
The Hole Thing

1. Suppose that a cylindrical hole 6 *cm* long has been drilled straight through the center of a solid sphere. What is the volume remaining in (what used to be) the sphere?

Solution 1. *(A clever way ...)* If the question is being asked at all, there ought to be only one possible answer, no matter what the size of the sphere is. (Except that it must have a diameter at least 6 *cm*, of course.) Hence we may choose a sphere of a convenient size, namely a diameter of 6 *cm*, in which case the hole drilled through the sphere would have to be infinitely thin and be a diameter. The volume left over after an infinitely thin hole is drilled through this sphere (of radius $\frac{6}{2} = 3\ cm$) is $V = \frac{4}{3}\pi 3^3 = 4\pi 9 = 36\pi\ cm^3$.

The problem with this solution is that one really needs a better reason why "there ought to be only one possible answer." (For full credit, anyway!) ■

Solution 2. *(A conventional approach ...)* We will set up the solid that remains after the hole is drilled through a sphere of radius r (where $r \geq 3$, of course) as a solid of revolution and then compute its volume using the washer method.



A sphere of radius r can be obtained by rotating the right half of the circle $x^2+y^2 = r^2$ about the y-axis. Note that $y = \pm 3$ on the right half of this circle when $x = \sqrt{r^2 - 3^2} = \sqrt{r^2 - 9}$. The region we will rotate about the y-axis to get the solid we want has as its left border the line $x = \sqrt{r^2 - 3^2}$ and as its right border (an arc of) the circle $x^2 + y^2 = r^2$. Note that the hole is then $3 - (-3) = 6\ cm$ long.

Since we rotated about the y-axis, washers for this solid will be stacked along the y-axis. It is not hard to check that the washer at y for this solid, where $-3 \leq y \leq 3$, has outer radius $S = x = \sqrt{r^2 - y^2}$ and inner radius $s = \sqrt{r^2 - 9}$. The volume of the solid is thus:

$$\begin{aligned}\int_{-3}^{3} \pi \left(S^2 - s^2\right)\, dy &= \pi \int_{-3}^{3} \left(\left[\sqrt{r^2 - y^2}\right]^2 - \left[\sqrt{r^2 - 9}\right]^2\right)\, dy \\ &= \pi \int_{-3}^{3} \left(\left[r^2 - y^2\right] - \left[r^2 - 9\right]\right)\, dy = \pi \int_{-3}^{3} \left(9 - y^2\right)\, dy \\ &= \pi \left(9y - \frac{1}{3}y^3\right)\Big|_{-3}^{3} = \pi \left[\left(9 \cdot 3 - \frac{1}{3}3^3\right) - \left(9 \cdot (-3) - \frac{1}{3}(-3)^3\right)\right] \\ &= \pi \left(2 \cdot 27 - 2 \cdot 9\right) = \pi(54 - 18) = 36\pi\ cm^3 \qquad \blacksquare\end{aligned}$$