Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2024
Solutions to the Final Examination
11:00-14:00 on Saturday, 13 April, in the Gym.

Time: *3 hours.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts **A**, **B**, and **C**, and, if you wish, part **D**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: ~~Open book~~ aid sheet, most any calculator, one head-mounted neural net.

Part A. Do all four (4) of **1**–**4**.

1. Evaluate any *four* (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

$$\textbf{a. } \int_0^\infty \frac{1}{(x+2)^3}\,dx \quad \textbf{b. } \int 4xe^{x^2+1}\,dx \quad \textbf{c. } \int_0^{\pi/2} \sin^{17}(x)\cos(x)\,dx$$

$$\textbf{d. } \int \frac{1}{x^2-1}\,dx \quad \textbf{e. } \int_1^e \ln(x)\,dx \quad \textbf{f. } \int \frac{1}{4-x^2}\,dx$$

Solutions. **a.** Since we have ∞ as one of the limits, this is an improper integral and should be evaluated using a limit. Along the way we will use the substitution $w = x + 2$, so $dw = dx$, and change the limits accordingly: $\begin{matrix} x & 0 & t \\ w & 2 & t+2 \end{matrix}$ Then

$$\begin{aligned}\int_0^\infty \frac{1}{(x+2)^3}\,dx &= \lim_{t\to\infty}\int_0^t \frac{1}{(x+2)^3}\,dx = \lim_{t\to\infty}\int_2^{t+2}\frac{1}{w^3}\,dw = \lim_{t\to\infty}\int_2^{t+2} w^{-3}\,dw \\ &= \lim_{t\to\infty}\left.\frac{w^{-2}}{-2}\right|_2^{t+2} = \lim_{t\to\infty}\left.\frac{-1}{2w^2}\right|_2^{t+2} = \lim_{t\to\infty}\left(\frac{-1}{2(t+2)^2} - \frac{-1}{2\cdot 2^2}\right) \\ &= \lim_{t\to\infty}\left(\frac{-1}{2(t+2)^2} + \frac{1}{8}\right) = 0 + \frac{1}{8} = \frac{1}{8} = 0.125,\end{aligned}$$

since $2(t+2)^2 \to \infty$ as $t \to \infty$. □

b. We will use the substitution $u = x^2 + 1$, so $du = 2x\,dx$ and $4x\,dx = 2\,du$. Then:

$$\int 4xe^{x^2+1}\,dx = \int e^u\,2\,du = 2e^u + C = 2e^{x^2+1} + C \quad \square$$

c. We will use the substitution $z = \sin(x)$, so $dz = \cos(x)\,dx$, and change the limits as we go along: $\begin{matrix} x & 0 & \pi/2 \\ z & 0 & 1 \end{matrix}$

$$\int_0^{\pi/2}\sin^{17}(x)\cos(x)\,dx = \int_0^1 z^{17}\,dz = \left.\frac{z^{18}}{18}\right|_0^1 = \frac{1^{18}}{18} - \frac{0^{18}}{18} = \frac{1}{18} - 0 = \frac{1}{18} \quad \square$$

d. Observe that $\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$, so we will have to use partial fractions to decompose the integral:

$$\begin{aligned}\frac{1}{x^2-1} &= \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} \\ &= \frac{Ax+A+Bx-B}{(x-1)(x+1)} = \frac{(A+B)x+(A-B)}{(x-1)(x+1)}\end{aligned}$$

Comparing coefficients of powers of x in the numerators at the beginning and the end, we see that we must have $A+B=0$ and $A-B=1$. Adding these equations together gives us $2A=1$, so $A=\frac{1}{2}=0.5$, and plugging this back into either equation lets us solve for $B=-\frac{1}{2}=-0.5$. Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{1}{(x-1)(x+1)}\,dx = \int \frac{\frac{1}{2}}{x-1}\,dx + \int \frac{-\frac{1}{2}}{X+1}\,dx \\ &= \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx \qquad \text{Substitute } u=x-1 \text{ and } w=x+1, \text{ so } du=dx \text{ and } dw=dx. \\ &= \frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{2}\int \frac{1}{w}\,dw = \frac{1}{2}\ln(u) - \frac{1}{2}\ln(w) + C \\ &= \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) + C. \quad \square\end{aligned}$$

e. We will use integration by parts, with $u=\ln(x)$ and $v'=1$, so $u'=\frac{1}{x}$ and $v=x$. Then

$$\begin{aligned}\int_1^e \ln(x)\,dx &= x\ln(x)\big|_1^e - \int_1^e \frac{1}{x}\cdot x\,dx = e\ln(e) - 1\ln(1) - \int_1^e 1\,dx \\ &= e\cdot 1 - 1\cdot 0 - x\big|_1^e = e - 0 - (e-1) = e - e + 1 = 1 \quad \square\end{aligned}$$

f. We will use the trigonometric substitution $x=2\sin(t)$, so $dx=2\cos(t)\,dt$. Note that then $\sin(t)=\frac{x}{2}$ and $\cos(t)=\sqrt{1-\sin^2(t)}=\sqrt{1-\frac{x^2}{4}}$.

$$\begin{aligned}\int \frac{1}{4-x^2}\,dx &= \int \frac{1}{4-(2\sin(t))^2}\,2\cos(t)\,dt = \int \frac{2\cos(t)}{4-4\sin^2(t)}\,dt \\ &= \int \frac{2\cos(t)}{4\left(1-\sin^2(t)\right)}\,dt = \int \frac{2\cos(t)}{4\cos^2(t)}\,dt = \int \frac{1}{2\cos(t)}\,dt \\ &= \frac{1}{2}\int \sec(t)\,dt = \frac{1}{2}\ln\left(\sec(t)+\tan(t)\right) + C = \frac{1}{2}\ln\left(\frac{1}{\cos(t)} + \frac{\sin(t)}{\cos(t)}\right) + C \\ &= \frac{1}{2}\ln\left(\frac{1}{\sqrt{1-\frac{x^2}{4}}} + \frac{\frac{x}{2}}{\sqrt{1-\frac{x^2}{4}}}\right) + C\end{aligned}$$

... which you may simplify at your leisure. :-) ■

2. Determine whether the series converges in any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

$$\textbf{a.}\ \sum_{n=0}^{\infty} \frac{n\sqrt{n}}{n^3+1} \qquad \textbf{b.}\ \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln\left(n^2\right)} \qquad \textbf{c.}\ \sum_{n=0}^{\infty} \frac{n+1}{\pi^n}$$

$$\textbf{d.}\ \sum_{n=0}^{\infty} \frac{3^{n-1}}{(n+1)!} \qquad \textbf{e.}\ \sum_{n=1}^{\infty} \frac{\cos\left(n^2\right)}{n^2} \qquad \textbf{f.}\ \sum_{n=0}^{\infty} n^2 e^{-n}$$

SOLUTIONS. **a.** $\displaystyle\sum_{n=0}^{\infty} \frac{n\sqrt{n}}{n^3+1} = \sum_{n=0}^{\infty} \frac{n^{3/2}}{n^3+1}$ converges by the Generalized p-Test because it has $p = 3 - \frac{3}{2} = \frac{3}{2} > 1$. □

b. We will apply the Alternating Series Test.

i. $\displaystyle\lim_{n\to\infty} \left|\frac{(-1)^n}{\ln\left(n^2\right)}\right| = \lim_{n\to\infty} \frac{1}{\ln\left(n^2\right)} \begin{matrix}\to 1\\ \to \infty\end{matrix} = 0$, so $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{\ln\left(n^2\right)} = 0$ too.

ii. Since $\ln\left(n^2\right) > 0$ for all $n \geq 2$, $\dfrac{(-1)^n}{\ln\left(n^2\right)}$ alternates sign because $(-1)^n$ does.

iii. Since n^2 and $\ln(x)$ are both increasing functions, $\ln\left(n^2\right) < \ln\left((n+1)^2\right)$ for all $n \geq 2$. It follows that $\displaystyle\left|\frac{(-1)^n}{\ln\left(n^2\right)}\right| = \frac{1}{\ln\left(n^2\right)} > \frac{1}{\ln\left((n+1)^2\right)} = \left|\frac{(-1)^{n+1}}{\ln\left((n+1)^2\right)}\right|$.

Hence, by the Alternating Series Test, the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln\left(n^2\right)}$ converges. □

c. We will use the Ratio Test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)+1}{\pi^{n+1}}}{\frac{n+1}{\pi^n}}\right| = \lim_{n\to\infty} \left|\frac{n+2}{\pi^{n+1}} \cdot \frac{\pi^n}{n+1}\right| = \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \frac{1}{\pi}$$
$$= \frac{1}{\pi} \cdot \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{1}{\pi} \cdot \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{1}{n}} = \frac{1}{\pi} \cdot \frac{1+0}{1+0} = \frac{1}{\pi} \cdot 1 = \frac{1}{\pi} < 1$$

It follows by the Ratio Test that the series $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{\pi^n}$ converges. □

d. We will use the Ratio Test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{3^{(n+1)-1}}{((n+1)+1)!}}{\frac{3^{n-1}}{(n+1)!}}\right| = \lim_{n\to\infty} \left|\frac{3^n}{(n+2)!} \cdot \frac{(n+1)!}{3^{n-1}}\right|$$
$$= \lim_{n\to\infty} \frac{3}{n+2} \begin{matrix}\to 3\\ \to \infty\end{matrix} = 0 < 1$$

It follows by the Ratio Test that the series $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n-1}}{(n+1)!}$ converges □.

e. Observe that $0 \leq \left|\dfrac{\cos\left(n^2\right)}{n^2}\right| = \dfrac{\left|\cos\left(n^2\right)\right|}{n^2} \leq \dfrac{1}{n^2}$ since $|\cos(x)| \leq 1$ for all x. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Test, it follows that $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos\left(n^2\right)}{n^2}\right|$ converges by the Basic Comparison Test, from which it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(n^2\right)}{n^2}$ converges absolutely, and hence converges. □

f. We will use the Ratio Test.

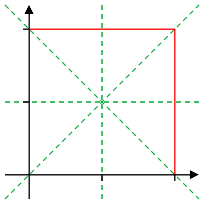
$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^2e^{-(n+1)}}{n^2e^{-n}}\right| = \lim_{n\to\infty}\frac{n^2+2n+1}{n^2}\cdot\frac{e^{-n-1}}{e^{-n}}\\ &= \lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)e^{-1} = \frac{1}{e}(1+0+0) = \frac{1}{e} < 1\end{aligned}$$

Thus $\displaystyle\sum_{n=0}^{\infty} n^2e^{-n}$ converges by the Ratio Test. ■

3. Do any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

- **a.** Find the centroid of the region above $y = 0$ and below $y = 2$ for $0 \leq x \leq 2$.
- **b.** Find the arc-length of the curve $y = x + 41$, where $0 \leq x \leq 4$.
- **c.** Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.
- **d.** Find the volume of the solid obtained by revolving the region between $y = x - 4$ and $y = 1$, where $4 \leq x \leq 5$, about the y-axis.
- **e.** Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-n)^n}{23^n}$ converges or diverges.
- **f.** Find the area of the finite region between $y = x$ and $y = x^4$.

SOLUTIONS. **a.** The region in question is the square with corners at $(0,0)$, $(2,0)$, $(0,2)$, and $(2,2)$. This has four lines of symmetry: $x = 1$, $y = 1$, $y = x$, and $y = 1 - x$. Since the centroid of a region must be on any line of symmetry of the region, it follows that the centroid of this region must be on the point where these four lines intersect, namely $(1,1)$. □



b. We plug $\dfrac{dy}{dx} = \dfrac{d}{dx}(x+41) = 1$ into the arc-length formula:

$$\begin{aligned}\text{arc-length} &= \int_0^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^4 \sqrt{1+1^2}\,dx = \int_0^4 \sqrt{2}\,dx\\ &= \sqrt{2}\cdot x\Big|_0^4 = \sqrt{2}\cdot 4 - \sqrt{2}\cdot 0 = 4\sqrt{2} \quad \square\end{aligned}$$

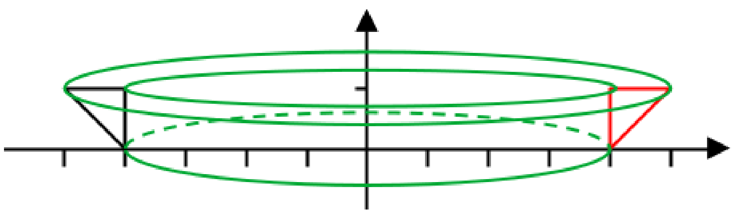
c. Note that $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. The partial fraction tricks we use to help integrate rational functions tell us that for some constants A and B we have

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1)+Bn}{n(n+1)} = \frac{(A+B)n+A}{n(n+1)}.$$

Comparing coefficients of n in the numerators at the beginning and end tells us that $A+B=0$ and $A=1$, so $B=-1$. It follows that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2+n} &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \cdots \\ &= 1 + \left[-\frac{1}{2} + \frac{1}{2}\right] + \left[-\frac{1}{3} + \frac{1}{3}\right] + \left[-\frac{1}{4} + \frac{1}{4}\right] + \cdots = 1+0+0+\cdots = 1. \quad \square\end{aligned}$$

d. Here is a sketch of the solid:



We give two solutions, using the disk/washer method and the cylindrical shell method, respectively.

i. Disk/washer method. The disks are perpendicular to the axis of rotation, *i.e.* the y-axis, so we use y as our variable. Note that $0 \le y \le 1$ over the given region. The disk at y has inner radius $r = 4 - 0 = 4$ and outer radius $R = x = y + 4$, since $y = x - 4$ on the right border of the region. It follows that:

$$\begin{aligned}V &= \int_0^1 \pi\left(R^2 - r^2\right)\, dy = \int_0^1 \pi\left((y+4)^2 - 4^2\right)\, dy = \int_0^1 \pi\left(y^2 + 8y + 16 - 16\right)\, dy \\ &= \int_0^1 \pi\left(y^2 + 8y\right)\, dy = \pi\left(\frac{y^3}{3} + \frac{8y^2}{2}\right)\Bigg|_0^1 = \pi\left(\frac{1^3}{3} + \frac{8\cdot 1^2}{2}\right) - \pi\left(\frac{0^3}{3} + \frac{8\cdot 0^2}{2}\right) \\ &= \pi\left(\frac{1}{3} + 4\right) - \pi \cdot 0 = \pi\left(\frac{1}{3} + \frac{12}{3}\right) - 0 = \frac{13\pi}{3} \approx 13.6136\end{aligned}$$

ii. Cylindrical shell method. The shells are parallel to the y-axis, so they are perpendicular to the x-axis, so we use x as our variable. Note that $4 \le x \le 5$ over the given region. The cylindrical shell at x has radius $r = x - 0 = x$ and height $R = 1 - y = 1 - (x-4) = 5 - x$, since $y = x - 4$ is the right border of the region. It follows that:

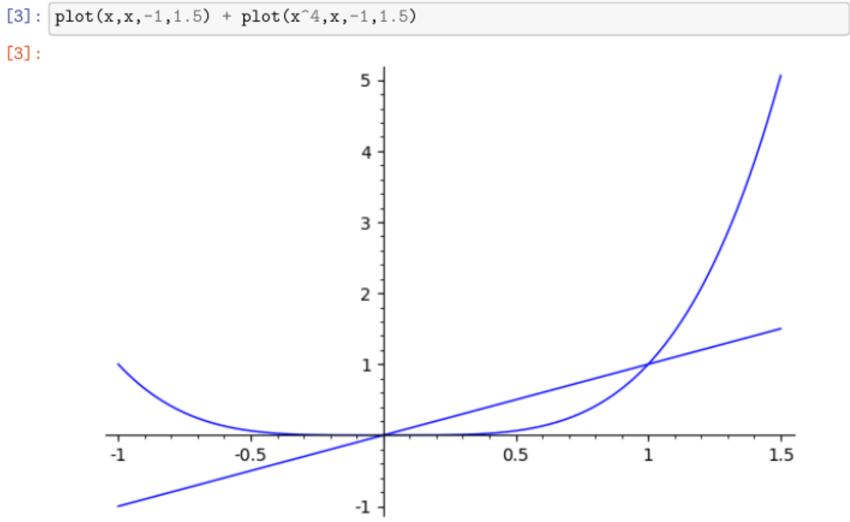
$$V = \int_4^5 2\pi r h\, dx = \int_4^5 2\pi x(5-x)\, dx = 2\pi \int_4^5 \left(5x - x^2\right)\, dx = 2\pi \left(\frac{5x^2}{2} - \frac{x^3}{3}\right)\Big|_4^5$$
$$= 2\pi\left(\frac{5\cdot 5^2}{2} - \frac{5^3}{3}\right) - 2\pi\left(\frac{5\cdot 4^2}{2} - \frac{4^3}{3}\right) = 2\pi\left(\frac{125}{2} - \frac{125}{3}\right) - 2\pi\left(\frac{80}{2} - \frac{64}{3}\right)$$
$$= 2\pi\cdot 125\cdot\left(\frac{1}{2} - \frac{1}{3}\right) - 2\pi\left(40 - 21 - \frac{1}{3}\right) = \frac{250\pi}{6} - 2\pi\frac{56}{3} = \frac{125\pi}{3} - \frac{112\pi}{3}$$
$$= \frac{13\pi}{3} \approx 13.6136 \quad \square$$

e. We will use the Root Test.

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{(-n)^n}{23^n}\right|^{1/n} = \lim_{n\to\infty} \left(\frac{n^n}{23^n}\right)^{1/n} = \lim_{n\to\infty} \frac{n \to \infty}{23 \to 23} = \infty > 1$$

It follows that $\displaystyle\sum_{n=0}^{\infty} \frac{(-n)^n}{23^n}$ diverges by the Root Test. $\square$

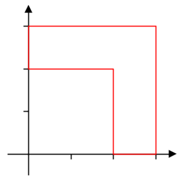
f. Here is how the curves $y = x$ and $y = x^4$ intersect.



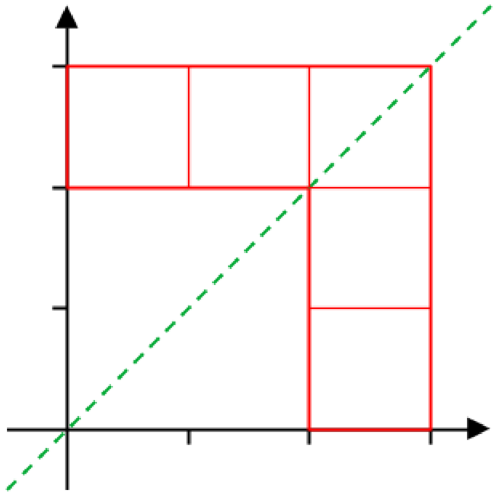
It's pretty obvious from the graph that the two curves intersect only when $x = 0$ and $x = 1$, and that in between $y = x$ is above $y = x^4$. One could also work this out algebraically: $x = x^4$ exactly when $x = 0$ or $x^3 = 1$, which last happens only when $x = 1$. Between $x = 0$ and $x = 1$, $y = x$ is above $y = x^4$ because $x^4 < x$ when $0 < x < 1$; for example, $\left(\frac{1}{2}\right)^4 = \frac{1}{16} < \frac{1}{2}$. It follows that the area of the region between the curves is:

$$A = \int_0^1 \left(x - x^4\right)\, dx = \left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^1 = \left(\frac{1^2}{2} - \frac{1^5}{5}\right) - \left(\frac{0^2}{2} - \frac{0^5}{5}\right)$$
$$= \left(\frac{1}{2} - \frac{1}{5}\right) - 0 = \frac{5}{10} - \frac{2}{10} = \frac{3}{10} \quad \blacksquare$$

4. Find the centroid of the "bent finger" region below $y = 3$ for $0 \le x \le 3$, and above $y = 2$ for $0 \le x \le 2$ but above $y = 0$ for $2 \le x \le 3$. *[12]*



SOLUTION. To help with the shortcuts used in this solution, consider the diagram below.



Observe that the line $y = x$ is a line of symmetry for the "bent finger" region, so the centroid must be on this line. This means that we only need to compute one of $\bar{x}$ or $\bar{y}$, since we must have $\bar{y} = \bar{x}$. Also, the region can be subdivided into five unit squares, so it has area $M = 5$. No need to do calculus to compute this! :-)

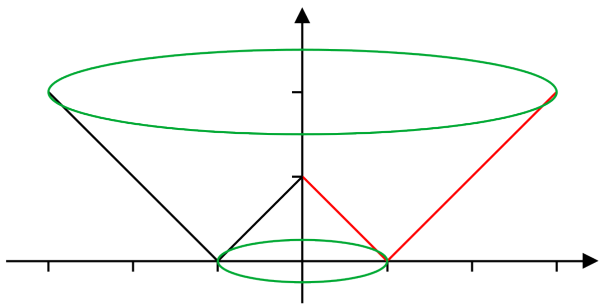
To compute $\bar{x}$ or $\bar{y}$, we still need to compute the moment M_y or M_x, respectively. We will compute M_y.

$$\begin{aligned} M_y &= \int_0^3 x \cdot [\text{length of vertical cross-section at } x]\, dx = \int_0^2 x(3-2)\, dx - \int_2^3 x(3-0)\, dx \\ &= \int_0^2 x\, dx + \int_2^3 3x\, dx = \left.\frac{x^2}{2}\right|_0^2 + \left.\frac{3x^2}{2}\right|_2^3 = \left(\frac{2^2}{2} - \frac{0^2}{2}\right) + \left(\frac{3\cdot 3^2}{2} - \frac{3\cdot 2^2}{2}\right) \\ &= \frac{4}{2} - 0 + \frac{27}{2} - \frac{12}{2} = \frac{19}{2} = 9.5 \end{aligned}$$

It follows that $\bar{x} = \dfrac{M_y}{M} = \dfrac{19/2}{5} = \dfrac{19}{10} = 1.9$. Since, as was previously noted, $\bar{x} = \bar{y}$ for this region, it follows that the centroid of this "bent finger" region has coordinates $(\bar{x}, \bar{y}) = \left(\dfrac{19}{10}, \dfrac{19}{10}\right) = (1.9, 1.9)$. ■

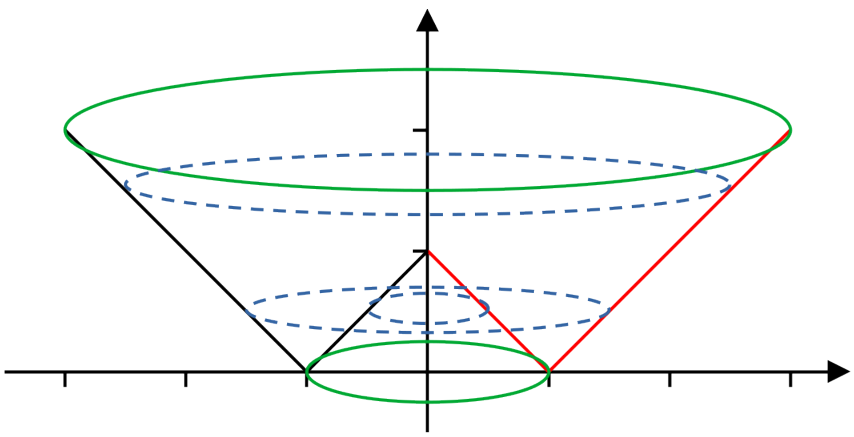
Part B. Do either *one* (1) of **5** or **6**. *[14]*

5. A solid is obtained by revolving the region below $y = 2$, and above $y = 1 - x$ for $0 \leq x \leq 1$ but above $y = x - 1$ for $1 \leq x \leq 3$, about the y-axis. Find the volume of this solid. *[14]*



SOLUTIONS. We give three solutions: one using the disk/washer method, one using the cylindrical shell method, and one using the formula for the volume of a cone.

i. Disk/washer method. Here is a sketch of the solid with a couple of disk/washer cross-sections drawn in:



The disks/washers are stacked vertically and are perpendicular to the axis of revolution, the y-axis, so we use y as our variable. Note that the original region has $0 \leq y \leq 2$. When $0 \leq y \leq 1$, the cross-section at y is a washer with outer radius $R = x = y + 1$ (since $y = x - 1$ on the right edge of the region) and inner radius $r = x = -y + 1$ (since $y = 1 - x$ on the left edge of the region for $0 \leq y \leq 1$), and when $1 \leq y \leq 2$, the cross-section at y is a disk with radius $R = x = y + 1$ (since $y = x - 1$ on the right edge of the region) and inner radius $r = x = 0$ (since $x = 0$, *i.e.* the y-axis, is the left edge of the region for $1 \leq y \leq 2$). It follows that volume of the solid is given by:

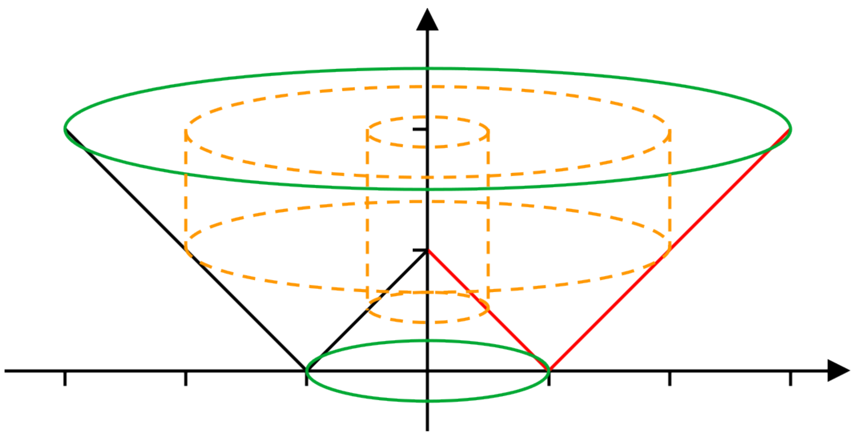
$$V = \int_0^2 \pi \left(R^2 - r^2\right) \, dy = \int_{)}^{1} \pi \left((y+1)^2 - (-y+1)^2\right) \, dy + \int_1^2 \pi \left((y+1)^2 - 0^2\right) \, dy$$

$$= \pi \int_0^1 \left(\left(y^2 + 2y + 1\right) - \left(y^2 - 2y + 1\right)\right) \, dy + \pi \int_1^2 \left(y^2 + 2y + 1\right) \, dy$$

$$= \pi \int_0^1 4y \, dy + \pi \int_1^2 \left(y^2 + 2y + 1\right) \, dy = \pi 2y^2\Big|_0^1 + \pi \left(\frac{y^3}{3} + y^2 + y\right)\Big|_1^2$$

$$= 2\pi \cdot 1^2 - 2\pi \cdot 0^2 + \pi \left(\frac{2^3}{3} + 2^2 + 2\right) - \pi \left(\frac{1^3}{3} + 1^2 + 1\right) = 2\pi + \frac{26}{3}\pi - \frac{7}{3}\pi = \frac{25\pi}{3} \quad \square$$

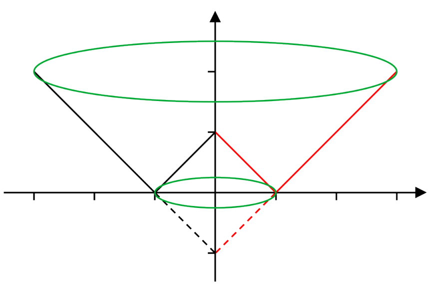
ii. Cylindrical shell method. Here is a sketch of the solid with a couple of cylindrical shells drawn in:



The cylindrical shells are parallel to the axis of revolution, the y-axis, and perpendicular to the x-axis, so we use x as our variable. The cylindrical shell at x has radius $r = x - 0 = x$ and height $h = 2 - (1 - x) = 1 + x$ when $0 \le x \le 1$, and has $r = x - 0 = x$ and height $h = 2 - (x - 1) = 3 - x$ for $1 \le x \le 3$. It follows that the volume of the solid is given by:

$$\begin{aligned} V &= \int_0^3 2\pi r h\, dx = \int_0^1 2\pi x(1+x)\, dx + \int_1^3 2\pi x(3-x)\, dx \\ &= 2\pi \int_0^1 \left(x^2 + x\right)\, dx + 2\pi \int_1^3 \left(-x^2 + 3x\right)\, dx \\ &= 2\pi \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^1 + 2\pi \left(-\frac{x^3}{3} + \frac{3x^2}{2}\right)\Big|_1^3 \\ &= 2\pi \left(\frac{1^3}{3} + \frac{1^2}{2}\right) - 2\pi \left(\frac{0^3}{3} + \frac{0^2}{2}\right) + 2\pi \left(-\frac{3^3}{3} + \frac{3 \cdot 3^2}{2}\right) - 2\pi \left(-\frac{1^3}{3} + \frac{3 \cdot 1^2}{2}\right) \\ &= 2\pi \frac{5}{6} - 2\pi 0 + 2\pi \frac{7}{6} + 2\pi \frac{27}{6} - 2\pi \frac{7}{6} = 2\pi \frac{25}{6} = \frac{25\pi}{3} \quad \square \end{aligned}$$

iii. Cone volume formula. The formula for the volume of a right circular cone with radius r at the flat end and with height h is $V = \dfrac{\pi r^2 h}{3}$. The solid in question can be thought of as a cone with radius 3 and height 3 ...



... from which two smaller cones, each of radius and height 1, have been removed. Thus the volume of the solid is $\dfrac{\pi 3^2 3}{3} - 2\dfrac{\pi 1^2 1}{3} = \dfrac{27\pi}{3} - \dfrac{2\pi}{3} = \dfrac{25\pi}{3}$. ■

6. Find the arc-length of the curve $y = \sqrt{4-x^2}$, where $0 \le x \le 2$,

a. using the arc-length formula and calculus *[10]*, and
b. without using the arc-length formula or calculus. *[4]*

SOLUTIONS. **a.** We plug

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\sqrt{4-x^2} = \frac{d}{dx}\left(4-x^2\right)^{1/2} = \frac{1}{2}\left(4-x^2\right)^{-1/2}\frac{d}{dx}\left(4-x^2\right) \\ &= \frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x) = -x\left(4-x^2\right)^{-1/2} = \frac{-x}{\sqrt{4-x^2}}\end{aligned}$$

and the fact that $0 \le x \le 2$ into the arc-length formula and integrate away:

$$\begin{aligned}\text{arc-length} &= \int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 \sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\,dx = \int_0^2 \sqrt{1+\frac{x^2}{4-x^2}}\,dx \\ &= \int_0^2 \sqrt{\frac{4-x^2}{4-x^2}+\frac{x^2}{4-x^2}}\,dx = \int_0^2 \sqrt{\frac{4}{4-x^2}}\,dx = \int_0^2 \frac{2}{\sqrt{4-x^2}}\,dx\end{aligned}$$

Substitute $x = 2\sin(\theta)$, so $dx = 2\cos(\theta)\,d\theta$, and change the limits as we go along: $\begin{matrix} x & 0 & 2 \\ \theta & 0 & \pi/2 \end{matrix}$

$$\begin{aligned}&= \int_0^{\pi/2} \frac{2}{\sqrt{4-4\sin^2(\theta)}}\,2\cos(\theta)\,d\theta = \int_0^{\pi/2} \frac{4\cos(\theta)}{\sqrt{4\left(1-\sin^2(\theta)\right)}}\,d\theta \\ &= \int_0^{\pi/2} \frac{4\cos(\theta)}{2\sqrt{\cos^2(\theta)}}\,d\theta = \int_0^{\pi/2} \frac{2\cos(\theta)}{\cos(\theta)}\,d\theta = \int_0^{\pi/2} 2\,d\theta = 2\theta\Big|_0^{\pi/2} \\ &= 2\cdot\frac{\pi}{2} - 2\cdot 0 = \pi \quad \square\end{aligned}$$

b. Observe that $y = \sqrt{4-x^2} \ge 0$ for $0 \le x \le 2$, and that

$$y = \sqrt{4-x^2} \Longrightarrow y^2 = 4-x^2 \Longrightarrow x^2+y^2 = 4.$$

This means the curve in question is the part of the circle of radius 2 centred at the origin for which $y \ge 0$ and $x \ge 0$, which is one quarter of the whole circle. The whole circle has circumference $2\pi r = 2\pi\cdot 2 = 4\pi$, one quarter of which – the arc-length of the curve in question – is π. ■

Part C. Do either *one* (1) of **7** or **8**. *[14]*

7. Find the Taylor series at 0 of $f(x) = e^{3x}$

a. using Taylor's formula, *[10]* and

b. without using Taylor's formula, at least directly. *[4]*

SOLUTIONS. **a.** We build the usual table to winkle out what $f^{(n)}(0)$ is in general. Note that $\frac{d}{dx}e^{3x} = e^{3x}\frac{d}{dx}3x = 3e^{3x}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{3x}	1
1	$3e^{3x}$	3
2	3^2e^{3x}	3^2
3	3^3e^{3x}	3^3
$\vdots$	$\vdots$	$\dots$

It's pretty easy to see that $f^{(n)}(0) = 3^n$ for $n \geq 0$. Plugging this into Taylor's formula tells us that the Taylor series of $f(x) = e^{3x}$ at 0 is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n = 1 + 3x + \frac{9}{2}x^2 + \frac{27}{6}x^3 + \cdots \quad \square$$

b. Recall from class or textbook that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ for all t. Substituting $t = 3x$ into this equation gives us $e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n$ for all x. Since a power series equal to a function must be the Taylor series of the function, it follows that the Taylor series at 0 of e^{3x} is $\sum_{n=0}^{\infty} \frac{3^n}{n!}x^n$. ■

8. Consider the power series $\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \cdots$.

a. Determine the radius and interval of convergence of this power series. *[6]*

b. What function has this power series as its Taylor series? *[4]*

c. What power series is equal to the product

$$\left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} (-x)^n\right) = \left(1 + x + x^2 + x^3 + \cdots\right)\left(1 - x + x^2 - x^3 + \cdots\right) \text{ ? } \textit{[4]}$$

SOLUTIONS. We give two solutions to each of parts **a** and **c**.

a. *By force of habit.* As usual, we apply the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{x^{2n}}\right| = \lim_{n\to\infty}\left|x^2\right| = \left|x^2\right|$$

It follows by the Ratio Test that the series converges (absolutely) when $\left|x^2\right| < 1$, *i.e.* when $-1 < x < 1$, and diverges when $\left|x^2\right| > 1$, *i.e.* when $x < 1$ or when $x > 1$. Thus the radius of convergence of this power series is $R = 1$.

It remains to determine whether the power series converges when $x = \pm 1$. Observe that for $x = \pm 1$, $x^{2n} = 1$. Hence, when $x = \pm 1$, $\lim_{n\to\infty} x^{2n} = \lim_{n\to\infty} 1 = 1 \neq 0$. It follows by the Divergence Test that the series diverges when $x = \pm 1$. Thus the interval of convergence of this powers series is $(-1, 1)$. □

a. *By recognition.* $\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \cdots$ is a geometric series with common ratio $r = x^2$. It follows that it converges exactly when $|r| = \left|x^2\right| < 1$, *i.e.* when $-1 < x < 1$, and diverges otherwise, so it has radius of convergence $R = 1$ and interval of convergence $(-1, 1)$. □

b. $\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \cdots$ is a geometric series with first term $a = 1$ and common ratio $r = x^2$. Using the summation formula for geometric series, it follows that $\sum_{n=0}^{\infty} x^{2n} = \frac{a}{1-r} = \frac{1}{1-x^2}$ when the series converges. When a power series is equal to a function, that power series is the Taylor series of the function, so $\sum_{n=0}^{\infty} x^{2n}$ is the Taylor series of the function $f(x) = \frac{1}{1-x^2}$. □

c. *Algebra!* We multiply the series out:

$$\begin{aligned}\left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty}(-x)^n\right) &= \left(1+x+x^2+x^3+\cdots\right)\left(1-x+x^2-x^3+\cdots\right) \\ &= \begin{array}{ccccccccccc} 1 & - & x & + & x^2 & - & x^3 & + & x^4 & - & \cdots \\ & + & x & - & x^2 & + & x^3 & - & x^4 & + & \cdots \\ & & & + & x^2 & - & x^3 & + & x^4 & - & \cdots \\ & & & & & + & x^3 & - & x^4 & - & \cdots \\ & & & & & & & \vdots & \vdots & \vdots & \vdots \end{array} \\ &= 1+x^2+x^4+\cdots = \sum_{n=0}^{\infty} x^{2n}. \quad \square\end{aligned}$$

c. *They're all geometric series!* Observe that $\sum_{n=0}^{\infty} x^n$ and $\sum_{n=0}^{\infty}(-x)^n$ are both geometric series, with common ratios of x and $-x$, respectively, and hence are equal to $\frac{1}{1-x}$ and $\frac{1}{1-(x)} = \frac{1}{1+x}$, respectively, when they converge. Using our solution to part **b**, it follows that

$$\left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty}(-x)^n\right) = \frac{1}{1-x}\cdot\frac{1}{1+x} = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}. \quad \blacksquare$$

[Total = 100]

Part D. Bonus problems! If you feel like it and have the time, do one or both of these.

3². Show that $\ln(\sec(x)-\tan(x)) = -\ln(\sec(x)+\tan(x))$. *[1]*

SOLUTION. Dream on! Also, tinker with how $\sec(x)$ and $\tan(x)$ are related in ways similar to computing $\int \sec(x)\,dx$. $\blacksquare$

2 × 5. Write a haiku (or several :-) touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

ENJOY YOUR SUMMER!

P.S.: You can keep this question sheet. (Souvenir, paper airplane, fire starter, the possibilities are endless! :-) The solutions to this exam will be posted to the course archive page at `http://euclid.trentu.ca/math/sb/1120H/` *in late April or early May.*