Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2024

Assignment #5
The Gamma Function
Due just before midnight on Friday, 16 February.*

Consider the Gamma function, the function of x defined by using x as a constant in an integral as follows:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt = \lim_{k\to\infty} \int_0^k t^{x-1}e^{-t}\,dt$$

This definition turns out to make sense whenever $x > 0$.

1. Use SageMath to compute $\Gamma\left(\frac{1}{2}\right)$, $\Gamma(1)$, $\Gamma\left(\frac{3}{2}\right)$, $\Gamma(2)$, $\Gamma\left(\frac{5}{2}\right)$, $\Gamma(3)$, $\Gamma\left(\frac{7}{2}\right)$, and $\Gamma(4)$. *[4]*

2. By hand, show that $\Gamma(x+1) = x\Gamma(x)$. *[4]*

You may have seen this before, but in case you haven't, $n!$, read as "n factorial", is defined for positive integers n as the product of all the positive integers less than or equal to n. That is, $n! = n(n-1)(n-2)\cdots 2\cdot 1$. To make verious formulas in various parts of mathematics work nicely without having to make exceptions, $0!$ is defined to be 1, *i.e.* $0! = 1$.

3. Using the results of questions **1** and **2**, explain why $\Gamma(n+1) = n!$ for any integer $n \geq 0$. *[2]*

NOTE: There are some very different ways to define the Gamma function. For example, it can be defined using an infinite product,

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{n=1}^{\infty} \frac{e^{x/n}}{1+\frac{x}{n}} = \frac{e^{-\gamma x}}{x} \cdot \frac{e^{x/1}}{1+\frac{x}{1}} \cdot \frac{e^{x/2}}{1+\frac{x}{2}} \cdot \frac{e^{x/3}}{1+\frac{x}{3}} \cdots,$$

where $\gamma = \lim_{k\to\infty} \left[\left(\sum_{n=1}^{k} \frac{1}{n} \right) - \ln(k+1) \right]$ is the constant we encountered in Assignment #4. This may be one reason it turns up in all sorts of places in mathematics, including applied mathematics, probability, and statistics.

The Gamma function also satisfies a lot of weird identities, such as $\Gamma(1-x)\Gamma(x) = \dfrac{\pi}{\sin(\pi x)}$ when $0 < x < 1$. Plugging $x = \frac{1}{2}$ into this identity is one way to get that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

---

* You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission via Blackboard fails, please submit your work to your instructor by email or on paper.