Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2024

Solutions to Assignment #5
The Gamma Function

Consider the Gamma function, the function of x defined by using x as a constant in an integral as follows:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt = \lim_{k\to\infty} \int_0^k t^{x-1}e^{-t}\,dt$$

This definition turns out to make sense whenever $x > 0$.

1. Use SageMath to compute $\Gamma\left(\frac{1}{2}\right)$, $\Gamma(1)$, $\Gamma\left(\frac{3}{2}\right)$, $\Gamma(2)$, $\Gamma\left(\frac{5}{2}\right)$, $\Gamma(3)$, $\Gamma\left(\frac{7}{2}\right)$, and $\Gamma(4)$. *[4]*

Solution. Here we go:

```
[1]: var('t')
     g = function('g')(x)
     assume(x>0)
     g(x) = integral( t^(x-1)*e^(-t), t, 0, oo)
     g(1/2)

[1]: sqrt(pi)

[2]: g(1)

[2]: 1

[3]: g(3/2)

[3]: 1/2*sqrt(pi)

[4]: g(2)

[4]: 1

[5]: g(5/2)

[5]: 3/4*sqrt(pi)

[6]: g(3)

[6]: 2

[7]: g(7/2)

[7]: 15/8*sqrt(pi)

[8]: g(4)

[8]: 6
```

□

Note: It is easy enough to define and use the Gamma function in SageMath, as was done above. The Gamma function is sufficiently important in mathematics that is actually built into SageMath as `gamma`. Typing `gamma(1/2)` into SageMath, for example, will give you `sqrt(pi)`, same as typing in `g(1/2)` did above.

2. By hand, show that $\Gamma(x+1) = x\Gamma(x)$. *[4]*

SOLUTION. We will use integration by parts at the key step. In what follows, we should assume $x > 0$, otherwise $\Gamma(x)$ may not be defined. Note also that as far as the variable of integration, t, is concerned, x is just some constant.

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^{(x+1)-1} e^{-t}\, dt = \int_0^\infty t^x e^{-t}\, dt \\
&= \lim_{k\to\infty} \int_0^k t^x e^{-t}\, dt \qquad \begin{matrix} u = t^x & v' = e^{-x} \\ u' = xt^{x-1} & v = (-1)e^{-t} \end{matrix} \\
&= \lim_{k\to\infty} \left[\left(-t^x e^{-t}\right)\Big|_0^k - \int_0^k (-1)xt^{x-1}e^{-t}\, dt \right] \\
&= \lim_{k\to\infty} \left[\left(-k^x e^{-k}\right) - \left(-0^x e^{-0}\right) + \int_0^k xt^{x-1}e^{-t}\, dt \right] \\
&= \lim_{k\to\infty} \left[\left(\frac{-k^x}{e^k}\right) - (-0\cdot 1) + \int_0^k xt^{x-1}e^{-t}\, dt \right] \\
&= \left[\lim_{k\to\infty} \frac{-k^x}{e^k} \right] - 0 + \left[\lim_{k\to\infty} x \int_0^\infty t^{x-1}e^{-t}\, dt \right] \\
&= 0 + x \lim_{k\to\infty} \int_0^\infty t^{x-1}e^{-t}\, dt \\
&= x\Gamma(x)
\end{aligned}$$

Note that $\lim_{k\to\infty} \frac{-k^x}{e^k} = 0$ since e^k dominates any power of k as $k \to \infty$. $\square$

You may have seen this before, but in case you haven't, $n!$, read as "n factorial", is defined for positive integers n as the product of all the positive integers less than or equal to n. That is, $n! = n(n-1)(n-2)\cdots 2\cdot 1$. To make verious formulas in various parts of mathematics work nicely without having to make exceptions, $0!$ is defined to be 1, *i.e.* $0! = 1$.

3. Using the results of questions **1** and **2**, explain why $\Gamma(n+1) = n!$ for any integer $n \geq 0$. *[2]*

SOLUTION. If $n = 0$, then $\Gamma(0+1) = \Gamma(1) = 1$, from the solution to question 1. If $n \geq 1$, then $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots 2\cdot 1\cdot \Gamma(1) = n!\cdot 1 = n!$. $\blacksquare$

NOTE: There are some very different ways to define the Gamma function. For example, it can be defined using an infinite product,

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{n=1}^{\infty} \frac{e^{x/n}}{1+\frac{x}{n}} = \frac{e^{-\gamma x}}{x} \cdot \frac{e^{x/1}}{1+\frac{x}{1}} \cdot \frac{e^{x/2}}{1+\frac{x}{2}} \cdot \frac{e^{x/3}}{1+\frac{x}{3}} \cdots,$$

where $\gamma = \lim_{k\to\infty} \left[\left(\sum_{n=1}^{k} \frac{1}{n} \right) - \ln(k+1) \right]$ is the constant we encountered in Assignment #4. This may be one reason it turns up in all sorts of places in mathematics, including applied mathematics, probability, and statistics.

The Gamma function also satisfies a lot of weird identities, such as $\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}$ when $0 < x < 1$. Plugging $x = \frac{1}{2}$ into this identity is one way to get that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.