Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2024

Solutions to Assignment #10
Series of Power

1. For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ converge? *[4]*

SOLUTION. Our first resort for such questions involving power series is the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^n x^{2n+1}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(-1)x^2}{(2n+3)(2n+2)}\right| = \lim_{n\to\infty}\frac{x^2}{(2n+3)(2n+2)}\begin{matrix}\to x^2\\ \to\infty\end{matrix} = 0 < 1$$

Note that x, and hence x^2, is a constant as far as n is concerned. Since the limit works out to be less than 1 no matter what real value x has, thes series converges by the Ratio Test for all x. □

2. What function does the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ equal when it converges? *[1]*

SOLUTION. Being lazy, we hand the problem off to SageMath.

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[2]: sum( (-1)^n*x^(2*n+1)/factorial(2*n+1), n, 0, oo)

[2]: sin(x)
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Thus $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$. □

3. For what values of x does the series $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$ converge? *[4]*

SOLUTION. Again, our first resort for such questions involving power series is the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{((n+1)+1)\,x^{n+1}}{(n+1)x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x}{n+1}\right| = |x|\cdot\lim_{n\to\infty}\frac{n+2}{n+1}$$
$$= |x|\cdot\lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = |x|\cdot\lim_{n\to\infty}\frac{1+\frac{2}{n}}{1+\frac{1}{n}} = |x|\cdot\frac{1+0}{1+0} = |x|$$

By the Ratio Test, it follows that the series converges when $|x| < 1$, *i.e.* when $= 1 < x < 1$, and diverges when $|x| > 1$, *i.e.* when $x <= 1$ or when $x > 1$. When $|x| = 1$, *i.e.* when $x = \pm 1$, the Ratio Test tells us nothing, so we have to check whether $\displaystyle\sum_{n=0}^{\infty}(n+1)(-1)^n$ and $\displaystyle\sum_{n=0}^{\infty}(n+1)1^n$ converge or diverge using some other test(s). Observe that

$$\lim_{n\to\infty}|(n+1)(-1)^n| = \lim_{n\to\infty}|(n+1)1^n| = \lim_{n\to\infty}(n+1) = \infty \neq 0,$$

so the series for both $x = -1$ and $x = 1$ diverge by the Divergence Test.

Thus $\sum_{n=0}^{\infty}(n+1)x^n$ converges when $-1 < x < 1$ and diverges when $x \leq -1$ or $x \geq 1$. $\square$

4. What function does the series $\sum_{n=0}^{\infty}(n+1)x^n$ equal when it converges? *[1]*

SOLUTION I. Being lazy, we hand the problem off to SageMath.

```
[1]: var('n')
     sum( (n+1)*x^n, n, 0, oo )

[1]: 1/(x^2 - 2*x + 1)
```

Thus $\sum_{n=0}^{\infty}(n+1)x^n = \dfrac{1}{x^2-2x+1}$. $\square$

SOLUTION II. Being clever, we observe that $\int (n+1)x^n\,dx = (n+1)\dfrac{x^{n+1}}{n+1} = x^{n+1}$ for each $n \geq 0$, at least up to some constant. It follows, at least when everything converges, that

$$\int \left[\sum_{n=0}^{\infty}(n+1)x^n\right] dx = \sum_{n=0}^{\infty}\int (n+1)x^n\,dx = C + \sum_{n=0} x^{n+1} = C + \sum_{k=1}^{\infty} x^k$$

for some constant C. This looks an awful lot like the geometric series $\sum_{k=0}^{\infty} x^k$ which we know sums to $\dfrac{1}{1-x}$ when it converges. Since

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{-1}{(1-x)^2}\cdot\frac{d}{dx}(1-x) = \frac{-1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2} = \frac{1}{x^2-2x+1}$$

and

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right) = \sum_{k=0}^{\infty}\frac{d}{dx}x^k = \sum_{k=0}^{\infty} kx^{k-1} = \sum_{n=0}^{\infty}(n+1)x^n,$$

it follows that $\sum_{n=0}^{\infty}(n+1)x^n = \dfrac{1}{x^2-2x+1}$. $\blacksquare$