Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2022

Assignment #8
Limits and sums for an odd shape
Due on Friday, 18 March.

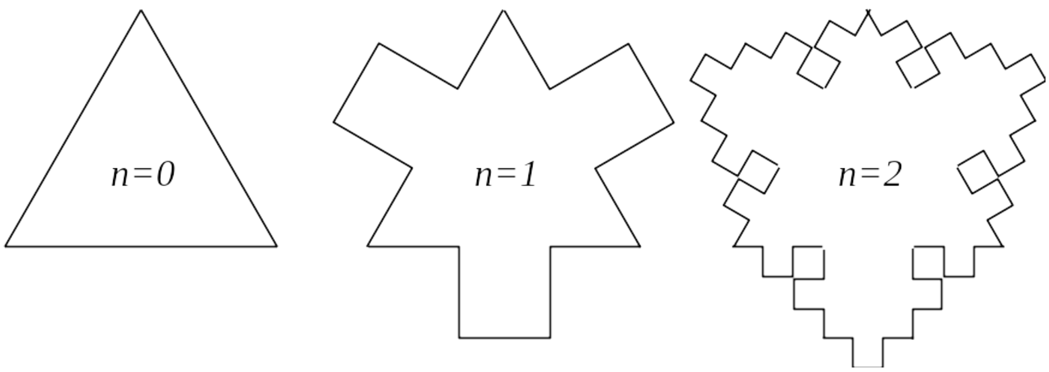
Please show all your work. As with all the assignments in this course, unless stated otherwise on the assignment, you are permitted to work together and look things up, so long as you acknowledge the sources you used and the people you worked with.

Consider the following process:

Step 0: In the beginning we have an equilateral triangle with sides of length 1.

Step $n+1$: Take the shape you have at the end of Step n and replace each straight segment of the perimeter by five segments of one-third the length, arranged to add a small square to the area of the old shape in the middle of the old line segment.

The diagram below shows you what you have at the end of Steps $n = 0$, 1, and 2.



1. Find (and justify!) expressions in terms of n for the length of perimeter and the area of the shape at the end of Step n. These expressions should work for all $n \geq 0$. *[6]*

Solution. First, the perimeter. Let p_n denote the length of the perimeter at (the end of) stage n.

0. At stage 0 we have an equilateral triangle with sides of length 1, so $p_0 = 3 \cdot 1 = 3$.
1. At stage 1 we replace each side of the equilateral triangle with 5 line segments of length $\frac{1}{3}$ each, so $p_1 = \frac{5}{3} \cdot p_0 = \frac{5}{3} \cdot 3 = 5$.
2. At stage 2 we replace each line segment making up the perimeter in p_1 with 5 line segments $\frac{1}{3}$ as long, so $p_2 = \frac{5}{3} \cdot p_1 = \frac{5}{3} \cdot \frac{5}{3} \cdot p_0 = \left(\frac{5}{3}\right)^2 \cdot 3 = \frac{25}{3}$.
3. At stage 3 we replace each line segment making up the perimeter in p_2 with 5 line segments $\frac{1}{3}$ as long, so $p_3 = \frac{5}{3} \cdot p_2 = \frac{5}{3} \cdot \left(\frac{5}{3}\right)^2 \cdot 3 = \left(\frac{5}{3}\right)^3 \cdot 3 = \frac{125}{9}$.

It is not hard to see the pattern developing here:

n At stage n we replace each line segment making up the perimeter in $p_{n-1}2$ with 5 line segments $\frac{1}{3}$ as long, so $p_n = \frac{5}{3} \cdot p_{n-1} = \frac{5}{3} \cdot \left(\frac{5}{3}\right)^{n-1} \cdot 3 = \left(\frac{5}{3}\right)^n \cdot 3 = \frac{5^n}{3^{n-1}}$.

Thus the length of the perimeter at (the end of) stage n is $p_n = \left(\dfrac{5}{3}\right)^n \cdot 3 = \dfrac{5^n}{3^{n-1}}$.

Second, let's tackle the area. Let a_n be the area of the shape at (the end of) stage n.

0. At stage 0 we have an equilateral triangle with 3 sides of length 1, which has a base of 1 and a height of $\frac{\sqrt{3}}{2}$ (Why?), and hence has area $a_0 = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$.
1. At stage 1 we add three squares, one for each side we had at stage 0, with side lengths equal to $\frac{1}{3}$ to the area from stage 0, so $a_1 = a_0 + 3 \cdot \left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{4} + 3 \cdot \frac{1}{9} = \frac{\sqrt{3}}{4} + \frac{1}{3}$. Note that at the end of stage 1 we have a perimeter consisting of $5 \cdot 3 = 15$ line segments, each of length $\frac{1}{3}$.
2. At stage 2 we add $5 \cdot 3 = 15$ squares, one for each line segment we have at the end of stage 1, with side lengths equal to $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$ to the area from stage 1, so $a_2 = a_1 + 5 \cdot 3 \cdot \left(\frac{1}{9}\right)^2 = \frac{\sqrt{3}}{4} + \frac{1}{3} + \frac{5}{3^3}$. Note that at the end of stage 2, we have a perimeter consisting of $5 \cdot 5 \cdot 3 = 5^2 \cdot 3$ line segments, each of length $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$.
3. At stage 3 we add $5^2 \cdot 3 = 75$ squares, one for each line segment we have at the end of stage 2, with side lengths equal to $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$ to the area from stage 2, so $a_3 = a_2 + 5^2 \cdot 3 \cdot \left(\frac{1}{27}\right)^2 = \frac{\sqrt{3}}{4} + \frac{1}{3} + \frac{5}{3^3} + \frac{5^2}{3^5}$. Note that at the end of stage 3, we have a perimeter consisting of $5 \cdot 5^2 \cdot 3 = 5^3 \cdot 3$ line segments, each of length $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$.

The pattern here is a little more complicated than the one for the length of the perimeter was, but it's still not too hard to get:

n. At stage n we add $5^{n-1} \cdot 3$ squares, one for each line segment we have at the end of stage $n-1$, with side lengths equal to $\frac{1}{3} \cdot \left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^n$ to the area from stage $n-1$, so $a_n = a_{n-1} + 5^{n-1} \cdot 3 \cdot \left(\frac{1}{3}\right)^{2n} = \frac{\sqrt{3}}{4} + \frac{1}{3} + \frac{5}{3^3} + \cdots + \frac{5^{n-1}}{3^{2n-1}}$. Note that at the end of stage n, we have a perimeter consisting of $5 \cdot 5^{n-1} \cdot 3 = 5^n \cdot 3$ line segments, each of length $\left(\frac{1}{3}\right)^n$.

Thus the area of the shape (at the end of) stage n is

$$a_n = \frac{\sqrt{3}}{4} + \frac{1}{3} + \frac{5}{3^3} + \cdots + \frac{5^{n-1}}{3^{2n-1}} = \frac{\sqrt{3}}{4} + \sum_{i=1}^{n} \frac{5^{i-1}}{3^{2i-1}} = \frac{\sqrt{3}}{4} + \sum_{k=0}^{n-1} \frac{1}{3} \cdot \left(\frac{5}{9}\right)^k .$$

Apart from the $\dfrac{\sqrt{3}}{4}$ at the beginning, this is a finite geometric series with first term $a = \frac{1}{3}$ and common ratio $r = \frac{5}{3^2} = \frac{5}{9}$, so we can use the formula for the sum of a finite geometric series: $a + ar + ar^2 + \cdots + ar^m = \frac{a\left(1-r^{m+1}\right)}{1-r}$. In this case, we have $m = n-1$, so

$$\begin{aligned} a_n &= \frac{\sqrt{3}}{4} + \sum_{k=0}^{n-1} \frac{1}{3} \cdot \left(\frac{5}{9}\right)^k = \frac{\sqrt{3}}{4} + \frac{\frac{1}{3}\left(1 - \left(\frac{5}{9}\right)^{n-1+1}\right)}{1 - \frac{5}{9}} \\ &= \frac{\sqrt{3}}{4} + \frac{1}{3} \cdot \frac{1 - \left(\frac{5}{9}\right)^n}{\frac{4}{9}} = \frac{\sqrt{3}}{4} + \frac{1}{3} \cdot \frac{9}{4} \cdot \left(1 - \left(\frac{5}{9}\right)^n\right) \\ &= \frac{\sqrt{3}}{4} + \frac{3}{4}\left(1 - \left(\frac{5}{9}\right)^n\right) . \end{aligned}$$

Thus the area of the shape at the end of stage n is $a_n = \dfrac{\sqrt{3}}{4} + \dfrac{3}{4}\left(1 - \left(\dfrac{5}{9}\right)^n\right)$. ■

2. Compute the length of the perimeter and the area of the shape that one gets as the limit of the process. *[3]*

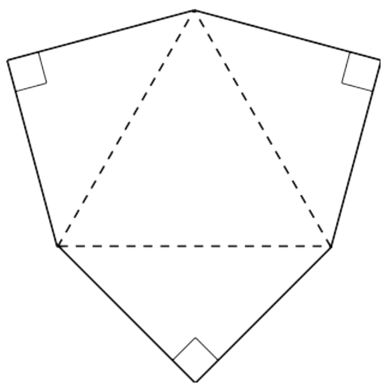
SOLUTION. We simply take the limits as $n \to \infty$ of the expressions for the length of the perimeter and area we have from doing **1**:

$$\begin{aligned}
\text{length} = \lim_{n\to\infty} p_n &= \lim_{n\to\infty} 3 \cdot \left(\frac{5}{3}\right)^n = \infty \quad \text{since } \frac{5}{3} > 1. \\
\text{area} = \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left[\frac{\sqrt{3}}{4} + \frac{3}{4}\left(1 - \left(\frac{5}{9}\right)^n\right)\right] \\
&= \frac{\sqrt{3}}{4} + \frac{3}{4}(1-0) = \frac{\sqrt{3}}{4} + \frac{3}{4} \quad \text{since } \frac{5}{9} < 1.
\end{aligned}$$

Thus the final shape has infinite perimeter while enclosing a finite area. A teeny, tiny, very little bit counterintuitive ... ■

3. Just what is the shape that one gets at the limit of the process? *[1]*

SOLUTION. It's almost a hexagon, specifically, the hexagon you get when you glue three right-angled isosceles $\frac{1}{\sqrt{2}}$-$\frac{1}{\sqrt{2}}$-1 triangles by their hypotenuses to the sides of an equilateral triangle with sides of length 1. Here's a sketch of this hexagon:



The shape we obtain at the end of this process has a much more complicated perimeter, which does touch the perimeter of the hexagon at infinitely many points (Why?), but also includes many points inside the hexagon (Again, why?), even though the hexagon and the ~~sheep~~ ~~shepe~~ shape we obtained have the same area.

The moral here is that "shape", "length", and "area" are pretty elusive concepts when you get down to details ... ■