Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2022

Solutions to Assignment #7
Optimization With Surface Area and Volume
Due on Friday, 11 March.

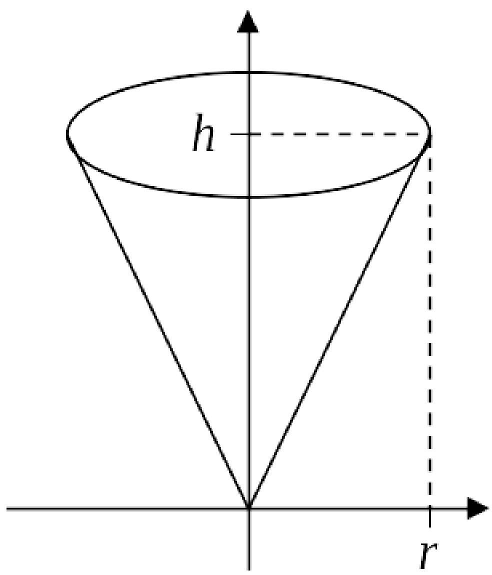
Please show all your work. As with all the assignments in this course, unless stated otherwise on the assignment, you are permitted to work together and look things up, so long as you acknowledge the sources you used and the people you worked with.

1. Suppose you are given a solid right circular cone with radius r at the blunt end and a height of h. Set up integrals to compute the

a. total surface area *[3]* and
b. volume *[3]*

of the cone, and then compute them using `SageMath`. (You may, if you wish, check your answers by computing the integrals by hand or just by looking up the relevant formulas.)

Solutions. In all that follows, we will set up the cone so that its tip is at the origin, pointing downwards, and with the y-axis as the cone's axis of symmetry, as in the sketch below.

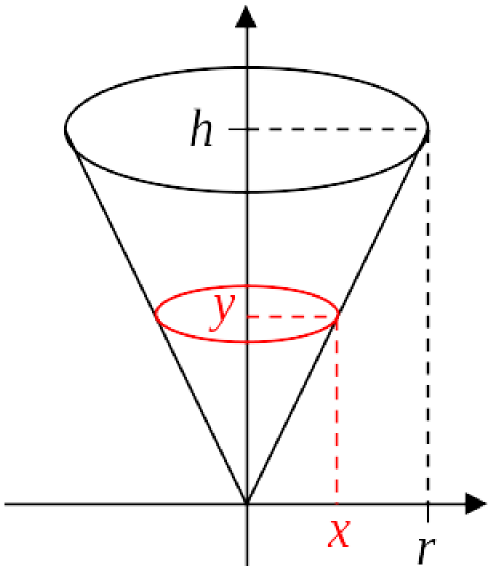


The right edge of the cone, that is, its intersection with the xy-plane, is the piece of the line $y = \frac{h}{r}x$ for which $0 \leq x \leq r$. It follows that the solid cone is obtained by revolving the region between the y-axis and $y = \frac{h}{r}x$, for $0 \leq x \leq r$ (or for $0 \leq y \leq h$), about the y-axis. Similarly, the surface of the cone is obtained by revolving the line $y = \frac{h}{r}x$, for $0 \leq x \leq r$, about the y-axis, to which we must add the blunt end of the cone, to get the complete surface.

a. If $y = \frac{h}{r}x$, then $\frac{dy}{dx} = \frac{h}{r}$, so our infinitesimal increment of arc-length works out to

$$\begin{aligned} ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{h}{r}\right)^2}\, dx \\ &= \sqrt{1 + \frac{h^2}{r^2}}\, dx = \sqrt{\frac{r^2 + h^2}{r^2}}\, dx = \frac{\sqrt{r^2 + h^2}}{r}\, dx\,. \end{aligned}$$

The diagram



makes it evident that when the piece of arc at x is revolved about the y-axis, it goes around a circle of radius $R = x - 0 = x$. Since $0 \le x \le r$ for the piece of the line $y = \frac{h}{r}x$ which gets revolved to make the cone, it follows that the surface area of the cone, not counting the blunt end, is given by:

$$\int_0^r 2\pi R\, ds = \int_0^r 2\pi x \frac{\sqrt{r^2 + h^2}}{r}\, dx = \int_0^r \frac{2\pi\sqrt{r^2 + h^2}}{r} x\, dx$$

Since this is just integrating a constant times x, you should be able to work this one out by hand pretty easily, but we were asked to do this using `SageMath`, so here goes:

```
sage:  var("r")
r
sage:  var("h")
h
sage:  var("y")
y
sage:  integral( 2*pi*sqrt(r^2 + h^2)/r*x, x, 0, r)
pi*sqrt(h^2 + r^2)*r
```

Note that the unknowns r, h, and y were all declared as variables, since they might all be used as such at some point in this `SageMath` session.

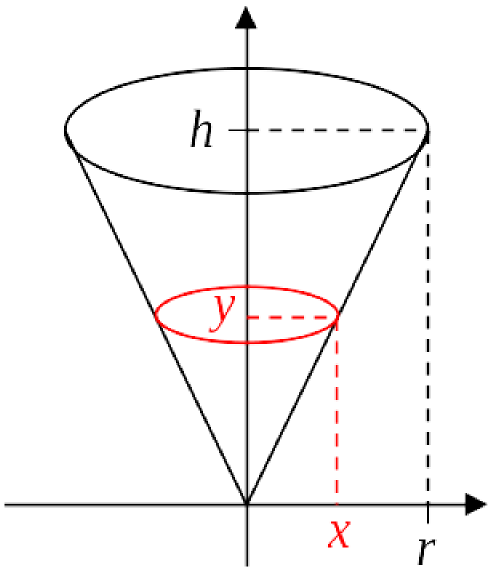
Anyway, this means that the surface area of the cone, not including the blunt end, is $\pi \cdot \sqrt{h^2 + r^2} \cdot r = \pi r \sqrt{r^2 + h^2}$. The blunt end of the cone is a circle of radius r that contributes πr^2 to the total surface area of the solid cone, so:

$$\text{Total surface area of the solid cone} = \pi r \sqrt{r^2 + h^2} + \pi r^2 = \pi r \left(r + \sqrt{r^2 + h^2} \right)$$

We could, of course, have set up a separate integral to compute the area of the blunt end, but you've probably seen that done before, so I skipped it! :-) □

QUESTION. As a small challenge, how would you work out the formula above without using calculus?

b. *Using the disk/washer method.* Since the axis of revolution is the y-axis using disks means that we should use y as the variable of integration because it is the y-axis that is perpendicular to the disks.



For the region being revolved to make the cone, *i.e.* the region between $y = 0$ and $y = \frac{h}{r}x$ for $0 \leq x \leq r$, we have $0 \leq y \leq h$. The disk at y has radius $R = x - 0 = x = \frac{r}{h}y$, so this disk has area $\pi R^2 = \pi \left(\frac{r}{h}y\right)^2 = \pi \frac{r^2}{h^2}y^2$. It follows that the volume of the come is given by:

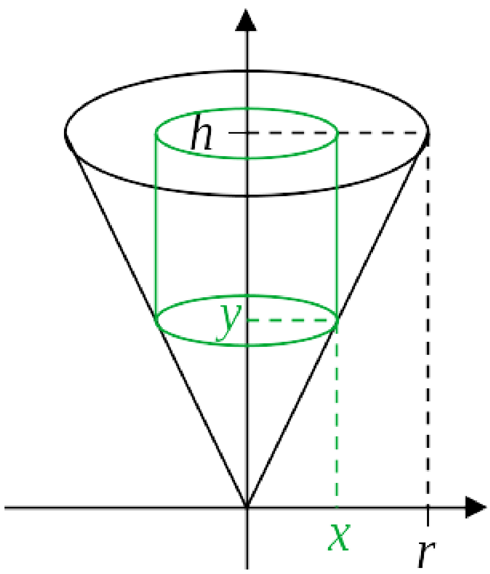
$$\text{Volume} = \int_0^h \pi R^2 \, dy = \int_0^h \frac{\pi r^2}{h^2} y^2 \, dy$$

We evaluate this integral using `SageMath`, continuing the session above:

```
sage:  integral( pi*r^2/h^2*y^2, y, 0, h )
1/3*pi*h*r^2
```

Thus the volume of the cone is $\frac{1}{3}\pi r^2 h$, which matches with what can be looked up or computed by other means. (See below!) □

b. *Using the cylindrical shell method.* Since the axis of revolution is the y-axis using cylindrical shells means that we should use x as the variable of integration because the x-axis is perpendicular to the shells while the y-axis is parallel to them.



As is evident from the diagram above, the shell at x, for some x with $0 \leq x \leq r$, has radius $R = x - 0 = x$ and height $H = h - y = h - \frac{h}{r}x = h\left(1 - \frac{x}{r}\right)$, and hence has area $2\pi RH = 2\pi xh\left(1 - \frac{x}{r}\right) = 2\pi h\left(x - \frac{x^2}{r}\right)$. It follows that the volume of the come is:

$$\text{Volume} = \int_0^r 2\pi RH\,dx = \int_0^r 2\pi h\left(x - \frac{x^2}{r}\right)\,dx$$

We evaluate this integral using `SageMath`, continuing the session above:

```
sage:  integral( 2*pi*h*(x-x^2/r), x, 0, r )
1/3*pi*h*r^2
```

Thus the volume of the cone is $\frac{1}{3}\pi r^2 h$, which matches with what can be looked up or computed by other means. (See above!) ■

2. Determine the minimum possible ratio of the surface area to the volume of a right circular cone. *[3]*

Solution. Combining the results obtained in question **1**, we see that the ratio of surface area to volume for a right-circular cone with height h and radius r at the blunt end is:

$$\frac{\text{Surface Area}}{\text{Volume}} = \frac{\pi r\left(r + \sqrt{r^2+h^2}\right)}{\frac{1}{3}\pi r^2 h} = \frac{3\left(r + \sqrt{r^2+h^2}\right)}{rh} = 3\left(\frac{1}{h} + \sqrt{\frac{1}{h^2} + \frac{1}{r^2}}\right)$$

How can we minimize this ratio when we have two variables, r and h, floating around? The short answer is that we can't really do so all that easily with the tools of single-variable calculus, expecially since the proportion of surface area to volume for a cone changes with scale. For example, consider the following two cones. One cone has $r = h = 1$, so its ratio of surface area to volume is $3\left(\frac{1}{1} + \sqrt{\frac{1}{1^2} + \frac{1}{1^2}}\right) = 3\left(1 + \sqrt{2}\right) \approx 7.2426$. The second cone is just like the first, but twice as high and wide, *i.e.* with $r = h = 2$, so its ratio of surface area to volume is $3\left(\frac{1}{2} + \sqrt{\frac{1}{2^2} + \frac{1}{2^2}}\right) = 3\left(\frac{1}{2} + \sqrt{\frac{1}{2}}\right) \approx 3.6213$. The bigger the cone with a given proportion of r to h, the smaller will its ratio of surface area to volume be.

It is not hard to see that if $r \to \infty$ or $h \to \infty$, and the other doesn't head to zero, the ratio of surface area to volume of a right-circular cone, $3\left(\frac{1}{h} + \sqrt{\frac{1}{h^2} + \frac{1}{r^2}}\right) \to 0$. Similarly, if $r \to 0^+$ or $h \to 0^+$, and the other doesn't head to infinity, we have $3\left(\frac{1}{h} + \sqrt{\frac{1}{h^2} + \frac{1}{r^2}}\right) \to \infty$. It follows that there is, in general, no (achievable) minimum ratio of surface area to volume for right-circular cones, at least not without additional constraints, though the ratio can get arbitrarily close to 0.

The same is true for other solids, by the way, because the surface area of any given solid grows as the square of the linear dimensions and its volume grows as the cube of the linear dimensions, and cubes grow faster than squares. ■

3. Dream up a practical application of knowing the answer to **2**. *[1]*

Solution. Well, **2** has no useful answer, but there is a small hint in the next-to-last paragraph of the solution that there may be useful answers in particular situations. Suppose, for example, that you have been tasked with designing a conical coffee cup that is to have a capacity of $0.5\ L = 500\ cm^3$. You wish to minimize the surface area the coffee exposes so that it cools as slowly as possible. (The rate of heat loss is proportional to surface area here, other things being equal.) The volume of coffee in the cup is conical, since the that's the shape of the cup. The constraint on volume means that you can solve for h as a function of r from the volume equation:

$$V = 500 = \frac{\pi}{3} r^2 h \implies h = \frac{1500}{\pi r^2}$$

One can then plug this into the expression for surface area,

$$SA = \pi r\left(r + \sqrt{r^2 + h^2}\right) = \pi r^2 + \pi r \sqrt{r^2 + \left(\frac{1500}{\pi r^2}\right)^2} = \pi r^2 + \sqrt{\pi^2 r^4 + \frac{2250000}{r^2}}\,,$$

and apply the usual tricks for minimizing a function of one variable from MATH 1110H. It's not pretty, so I'll leave you (and `SageMath`) to it, if you're interested. :-) ■