Mathematics 1120H – Calculus II: Integrals and Series

TRENT UNIVERSITY, Winter 2022

Solutions to Assignment #6
Arc-length of a parametric curves

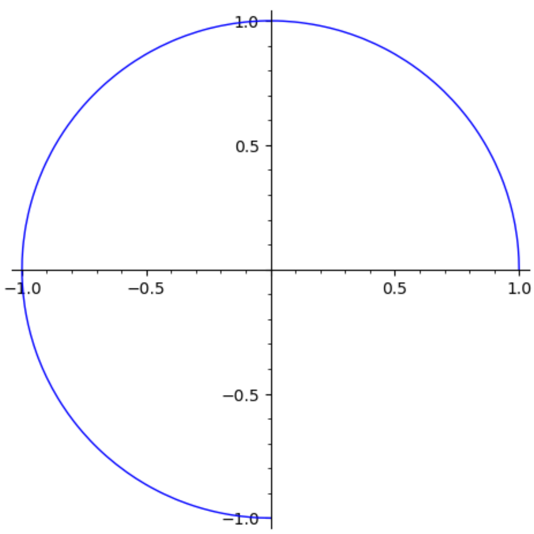
Due on Friday, 4 March.†

Please show all your work. As with all the assignments in this course, unless stated otherwise on the assignment, you are permitted to work together and look things up, so long as you acknowledge the sources you used and the people you worked with.

One way to describe or define a curve in two dimensions is by way of *parametric equations*, $x = f(t)$ and $y = g(t)$, where the x and y coordinates of points on the curve are simultaneously specified by plugging a third variable, called the *parameter* (in this case t), into functions $f(t)$ and $g(t)$. This approach can come in handy for situations where it is impossible to describe all of a curve as the graph of a function of x (or of y) and arises pretty naturally in various physics problems. (Think of specifying, say, the position (x, y) of a moving particle at time t.)

For a simple example, consider $x = \cos(t)$ and $y = \sin(t)$, for $0 \leq x \leq \frac{3\pi}{2}$. This gives three quarters of the unit circle centred at the origin, namely the parts of the unit circle that are in the first three quadrants. Here is a plot of the curve, as drawn by `SageMath`:

```
sage:  var("y")
y
sage:  var("t")
t
sage:  parametric_plot( (cos(t), sin(t)), (t,0,3*pi/2) )
```



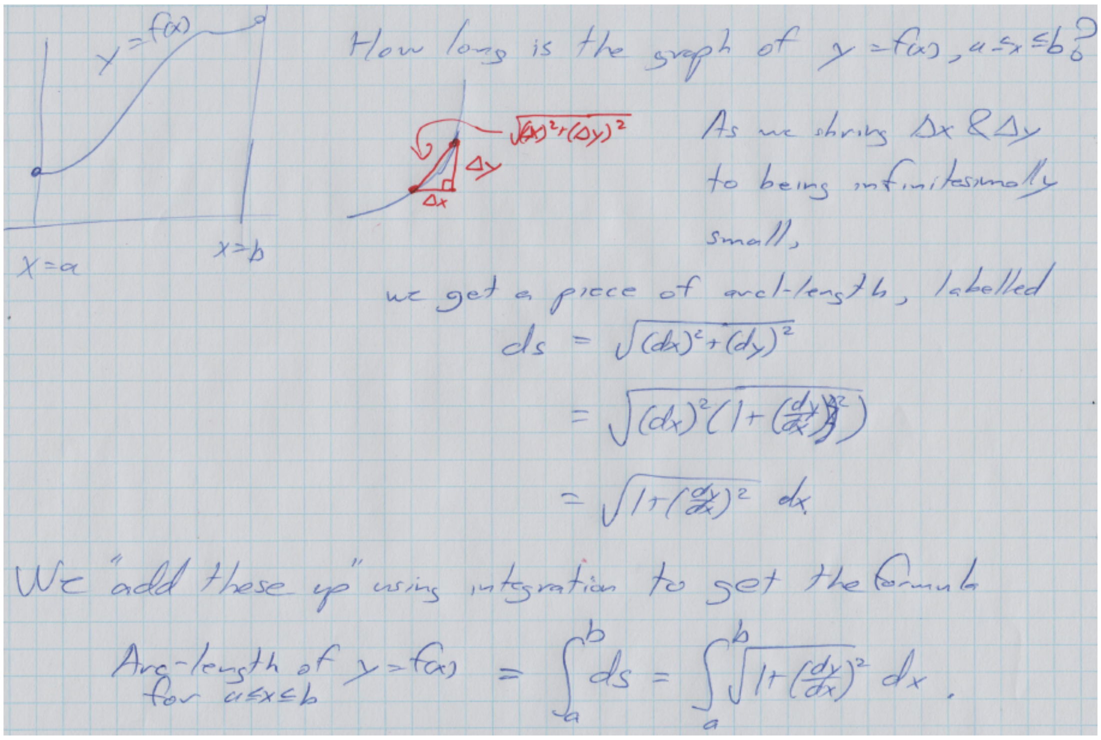
Note that this curve cannot be the graph of a single function of the form $y = f(x)$ because it fails the vertical line test, though it could be broken up into pieces which could each be so described.

† March Fo[u]rth is the only day of the year that doubles as a command! :-)

1. Suppose a parametric curve is given by $x = f(t)$ and $y = g(t)$, where $a \leq t \leq b$. Explain why the arc-length of this curve is given by $\displaystyle\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$. *[3]*

Hint: Reasoning similar to that given in the lecture on arc-length to justify why the arc-length of the curve $y = f(x)$, $c \leq x \leq d$, is $\displaystyle\int_c^d \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ will do the trick.

SOLUTION. Here's the reasoning in the lecture, minus the audio:



We can follow the same reasoning up until the definition of ds, at which point we diverge to take into account that $x = f(t)$ and $y = g(t)$, so $\dfrac{dx}{dt} = f'(t)$ and $\dfrac{dy}{dt} = g'(t)$, and put everything in terms of t:

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 + (dy)^2}\,\frac{dt}{dt} = \sqrt{[(dx)^2 + (dy)^2] \cdot \frac{1}{(dt)^2}}\, dt$$

$$= \sqrt{\frac{(dx)^2}{(dt)^2} + \frac{(dy)^2}{(dt)^2}}\, dt = \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt = \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Following the line of reasoning in the lecture, we now use integration to "add up" these infinitesimal increments of arc-length. Thus the arc-length of the curve given by $x = f(t)$ and $y = g(t)$, where $a \leq t \leq b$, is:

$$\text{Arc-length} = \int_a^b ds = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt \qquad \blacksquare$$

2. Use the arc-length formula for parametric curves from **1** to compute the length of the three-quarters of a circle given by $x = \cos(t)$ and $y = \sin(t)$, where $0 \leq x \leq \frac{3\pi}{2}$. Check your answer without using calculus! *[2]*

Solution. Since $f(t) = x = \cos(t)$ and $g(t) = y = \sin(t)$ for this curve, $f'(t) = \frac{d}{dt}\cos(t) = -\sin(t)$ and $g'(t) = \frac{d}{dt}\sin(t) = \cos(t)$. It follows that the length of the given curve is given by:

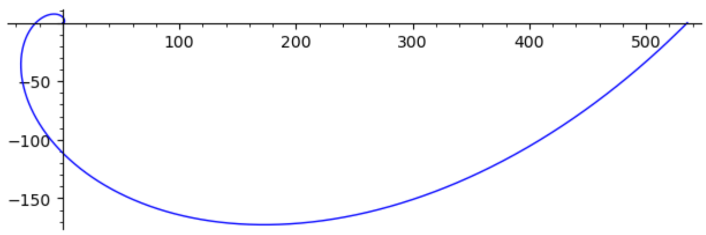
$$\begin{aligned}
\text{Arc-length} &= \int_0^{3\pi/2} \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_0^{3\pi/2} \sqrt{[-\sin(t)]^2 + [\cos(t)]^2}\,dt \\
&= \int_0^{3\pi/2} \sqrt{\sin^2(t) + \cos^2(t)}\,dt = \int_0^{3\pi/2} \sqrt{1}\,dt = \int_0^{3\pi/2} 1\,dt \\
&= t\Big|_0^{3\pi/2} = \frac{3\pi}{2} - 0 = \frac{3\pi}{2}
\end{aligned}$$

We can easily check this by recalling that the circumference of a complete circle of radius r is $2\pi r$. In this case, we have $r = 1$, so the radius of the complete unit circle is 2π. Three-quarters of that is $\frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}$, so the integral computation above got the right answer. ■

3. Plot the spiral given by $x = e^t\cos(t)$ and $y = e^t\sin(t)$, where $-2\pi \leq t \leq 2\pi$. You may use `SageMath` or other software to do so, if you want to. *[2]*

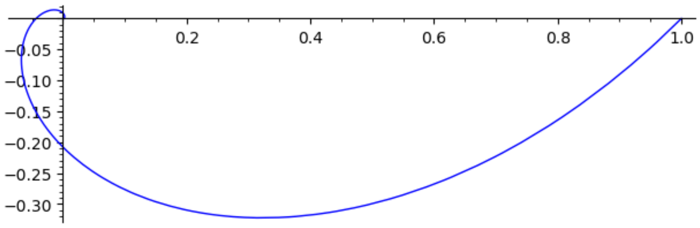
Solution. Following the example given on the first page of the assignment closely, we get:

```
sage: var("y")
y
sage: var("t")
t
sage: parametric_plot( (e^t*cos(t),e^t*sin(t)), (t,-2*pi,2*pi) )
```



The only problem here is that the scale gives us very little idea of what happens near the origin. The easiest way to deal with this is to limit the range of t to $-2\pi \leq t \leq 0$. Continuing the `SageMath` session above:

```
sage:  parametric_plot( (e^t*cos(t),e^t*sin(t)), (t,-2*pi,0) )
```



As you can see, the spiral homes in on the origin as t gets negative, though it never reaches it because $e^t > 0$ for all $t \in \mathbb{R}$, negative or not. ■

4. Use the arc-length formula for parametric curves from **1** to compute the length of the spiral given by $x = e^t \cos(t)$ and $y = e^t \sin(t)$, where $-2\pi \leq t \leq 2\pi$. *[3]*

Solution. Since $f(t) = x = e^t \cos(t)$ and $g(t) = y = e^t \sin(t)$ for this curve, $f'(t) = \frac{d}{dt} e^t \cos(t) = e^t \cos(t) - e^t \sin(t)$ and $g'(t) = \frac{d}{dt} e^t \sin(t) = e^t \sin(t) + e^t \cos(t)$. It follows that the length of the given curve is given by:

$$\text{Arc-length} = \int_{-2\pi}^{2\pi} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

$$= \int_{-2\pi}^{2\pi} \sqrt{[e^t \cos(t) - e^t \sin(t)]^2 + [e^t \sin(t) + e^t \cos(t)]^2}\, dt$$

$$= \int_{-2\pi}^{2\pi} \sqrt{(e^t)^2 \left[\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t)\right]}\, dt$$

$$= \int_{-2\pi}^{2\pi} \sqrt{(e^t)^2 \left[2\cos^2(t) + 2\sin^2(t)\right]}\, dt = \int_{-2\pi}^{2\pi} e^t \sqrt{2\left[\cos^2(t) + \sin^2(t)\right]}\, dt$$

$$= \int_{-2\pi}^{2\pi} e^t \sqrt{2 \cdot 1}\, dt = \sqrt{2} \int_{-2\pi}^{2\pi} e^t\, dt = \sqrt{2} \cdot e^t \Big|_{-2\pi}^{2\pi} = \sqrt{2} \cdot e^{2\pi} - \sqrt{2} \cdot e^{-2\pi}$$

$$= \sqrt{2}\left(e^{2\pi} - e^{-2\pi}\right) \approx 757.2969 \qquad \blacksquare$$