Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2022

Solutions to Assignment #4
A non-trigonometric integral reduction formula

Please show all your work. As with all the assignments in this course, unless stated otherwise on the assignment, you are permitted to work together and look things up, so long as you acknowledge the sources you used and the people you worked with.

1. Suppose $n > 1$. Find an integral reduction formula for $\displaystyle\int \frac{1}{(x^2+1)^n}\,dx$ *with* the help of a suitable trigonometric substitution. *[5]*

SOLUTION. We will use the trigonometric substitution $x = \tan(\theta)$, so $dx = \sec^2(\theta)\,d\theta$. We will also make use of the fact that $\sec(\theta) = \dfrac{1}{\cos(\theta}$, so $\cos(\theta) = \dfrac{1}{\sec(\theta)}$, and exploit the integral reduction formula for $\cos(\theta)$, namely:

$$\int \cos^k(\theta)\,dx = \frac{1}{k}\cos^{k-1}(\theta)\sin(\theta) + \frac{k-1}{k}\int \cos^{k-2}(\theta)\,d\theta$$

Here we go:

$$\begin{aligned}
\int \frac{1}{(x^2+1)^n}\,dx &= \int \frac{1}{\left(\tan^2(\theta)+1\right)^n}\sec^2(\theta)\,d\theta = \int \frac{1}{\left(\sec^2(\theta)\right)^n}\sec^2(\theta)\,d\theta \\
&= \int \frac{\sec^2(\theta)}{\sec^{2n}(\theta)}\,d\theta = \int \frac{1}{\sec^{2n-2}(\theta)}\,d\theta = \int \left(\frac{1}{\sec(\theta}\right)^{2n-2}\,d\theta \\
\text{Letting } k = 2n-2: \quad &= \int \cos^{2n-2}(\theta)\,d\theta = \frac{1}{2n-2}\cos^{2n-3}(\theta)\sin(\theta) + \frac{2n-3}{2n-2}\int \cos^{2n-4}(\theta)\,d\theta \\
&= \frac{1}{2n-2}\cos^{2n-3}(\theta)\frac{\cos(\theta)}{\cos(\theta)}\sin(\theta + \frac{2n-3}{2n-2}\int \left(\frac{1}{\sec(\theta)}\right)^{2n-4}(\theta)\,d\theta \\
&= \frac{1}{2n-2}\cos^{2n-2}(\theta)\frac{\sin(\theta)}{\cos(\theta)} + \frac{2n-3}{2n-2}\int \frac{1}{sec^{2n-4}(\theta)}\cdot\frac{\sec^2(\theta)}{\sec^2(\theta)}\,d\theta \\
&= \frac{1}{2n-2}\left(\frac{1}{\sec(\theta)}\right)^{2n-2}\tan(\theta) + \frac{2n-3}{2n-2}\int \frac{1}{\sec^{2n-2}(\theta)}\sec^2(\theta)\,d\theta \\
&= \frac{1}{2n-2}\cdot\frac{1}{\sec^{2n-2}(\theta)}\tan(\theta) + \frac{2n-3}{2n-2}\int \frac{1}{\left(\sec^2(\theta)\right)^{n-1}}\sec^2(\theta)\,d\theta \\
&= \frac{1}{2n-2}\cdot\frac{1}{\left(\sec^2(\theta)\right)^{n-1}}\tan(\theta) + \frac{2n-3}{2n-2}\int \frac{1}{\left(\tan^2(\theta)+1\right)^{n-1}}\sec^2(\theta)\,d\theta \\
&= \frac{1}{2n-2}\cdot\frac{\tan(\theta)}{\left(\tan^2(\theta)+1\right)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{1}{\left(\tan^2(\theta)+1\right)^{n-1}}\sec^2(\theta)\,d\theta \\
\text{Undoing the substitution:} \quad &= \frac{1}{2n-2}\cdot\frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{1}{(x^2+1)^{n-1}}\,dx
\end{aligned}$$

This gives us the following integral reduction formula, which works as long as $n > 1$:

$$\int \frac{1}{(x^2+1)^n}\,dx = \frac{1}{2n-2} \cdot \frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{1}{(x^2+1)^{n-1}}\,dx \qquad \blacksquare$$

2. Suppose $n > 1$. Find the same integral reduction formula for $\displaystyle\int \frac{1}{(x^2+1)^n}\,dx$ *without* using any trigonometric substitution. *[5]*

Solution. Instead of using a trigonometric substitution, as in the solution to question **1**, we will use integration by parts with the "dummy product" trick. We put the entire integrand into u, *i.e.* $u = \dfrac{1}{(x^2+1)^n}$, and let $v' = 1$, so $u' = \dfrac{-2nx}{(x^2+1)^{n+1}}$ and $v' = x$. We will also use some algebraic trickery.

$$\begin{aligned}
\int \frac{1}{(x^2+1)^n}\,dx &= \frac{1}{(x^2+1)^n} \cdot x - \int \frac{-2nx}{(x^2+1)^{n+1}} \cdot x\,dx \\
&= \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2}{(x^2+1)^{n+1}}\,dx \\
&= \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2+1-1}{(x^2+1)^{n+1}}\,dx \\
&= \frac{x}{(x^2+1)^n} + 2n \int \frac{x^2+1}{(x^2+1)^{n+1}}\,dx - 2n \int \frac{1}{(x^2+1)^{n+1}}\,dx \\
&= \frac{x}{(x^2+1)^n} + 2n \int \frac{1}{(x^2+1)^n}\,dx - 2n \int \frac{1}{(x^2+1)^{n+1}}\,dx
\end{aligned}$$

We can rearrange this and solve for $\displaystyle\int \frac{1}{(x^2+1)^n}\,dx$, as follows:

$$\begin{aligned}
(1-2n) \int \frac{1}{(x^2+1)^n}\,dx &= \frac{x}{(x^2+1)^n} - 2n \int \frac{1}{(x^2+1)^{n+1}}\,dx \\
\implies \quad \int \frac{1}{(x^2+1)^n}\,dx &= \frac{1}{1-2n} \cdot \frac{x}{(x^2+1)^n} - \frac{2n}{1-2n} \int \frac{1}{(x^2+1)^{n+1}}\,dx
\end{aligned}$$

Unfortunately, when we do this, the integral on the right-hand side actually has a power greater than n in it, not one that is less, so we have derived an anti-reduction formula! Not all is lost, however. If we let $k = n+1$, so $n = k-1$, the formula we obtained becomes:

$$\begin{aligned}
\int \frac{1}{(x^2+1)^{k-1}}\,dx &= \frac{1}{1-2(k-1)} \cdot \frac{x}{(x^2+1)^{k-1}} - \frac{2(k-1)}{1-2(k-1)} \int \frac{1}{(x^2+1)^k}\,dx \\
&= \frac{1}{3-2k} \cdot \frac{x}{(x^2+1)^{k-1}} - \frac{2k-2}{3-2k} \int \frac{1}{(x^2+1)^k}\,dx \\
&= \frac{-1}{2k-3} \cdot \frac{x}{(x^2+1)^{k-1}} + \frac{2k-2}{2k-3} \int \frac{1}{(x^2+1)^k}\,dx
\end{aligned}$$

If we solve this formula for $\displaystyle\int \frac{1}{(x^2+1)^k}\,dx$, we get:

$$\begin{aligned}\int \frac{1}{(x^2+1)^k}\,dx &= \frac{2k-3}{2k-2}\left[\int \frac{1}{(x^2+1)^{k-1}}\,dx - \frac{-1}{2k-3}\cdot\frac{x}{(x^2+1)^{k-1}}\right]\\ &= \frac{1}{2k-2}\cdot\frac{x}{(x^2+1)^{k-1}} + \frac{2k-3}{2k-2}\int \frac{1}{(x^2+1)^{k-1}}\,dx\end{aligned}$$

This is the same formula that we obtained in solving question **1**, except for using k in place of n, so we're done! ■

NOTE: Please do not use `SageMath`, except to check your answers.

AFTERWORD: Out of curiosity, let's see what `SageMath` does if we ask it to integrate $\displaystyle\int \frac{1}{(x^2+1)^n}\,dx$.

```
sage:  var("n")
n
sage:  integral(1/(x^2+1)^n,x)
x*hypergeometric((1/2, n), (3/2,), -x^2)
```

It seems that we get an answer in terms of a type of power series, namely "hypergeometric", with certain parameters. This *might* make more sense to us at the end of this course, once we've covered the material on power, and especially Taylor, series.