Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2022

Assignment #2
Let's integrate!

Due on Friday, 28 January. (May be submitted on paper or via Blackboard.)*

Please show all your work. As with all the assignments in this course, unless stated otherwise on the assignment, you are permitted to work together and look things up, so long as you acknowledge the sources you used and the people you worked with.

1. Compute the indefinite integral $\displaystyle\int \sin(\ln(x))\,dx$. *[3]*

Solution. We will integration by parts twice, each time using the same "dummy product" trick used to integrate $\ln(x)$ and $\arctan(x)$, which will get us to a multiple of the integral we started with, and then solve for it.

$$\begin{aligned}
\int \sin(\ln(x))\,dx &= x\sin(\ln(x)) - \int \cos(\ln(x))\cdot\frac{1}{x}\cdot x\,dx \\
&\qquad \left[\text{Using } u=\sin(\ln(x)) \text{ and } v'=1, \text{ so } u'=\cos(\ln(x))\cdot\frac{1}{x} \text{ and } v=x.\right] \\
&= x\sin(\ln(x)) - \int \cos(\ln(x))\,dx \\
&= x\sin(\ln(x)) - \left[x\cos(\ln(x)) - \int (-1)\sin(\ln(x))\cdot\frac{1}{x}\cdot x\,dx\right] \\
&\qquad \left[\text{Using } u=\cos(\ln(x)) \text{ and } v'=1, \text{ so } u'=-\sin(\ln(x))\cdot\frac{1}{x} \text{ and } v=x.\right] \\
&= x\sin(\ln(x)) - \left[x\cos(\ln(x)) + \int \sin(\ln(x))\,dx\right] \\
&= x\sin(\ln(x)) - x\cos(\ln(x)) - \int \sin(\ln(x))\,dx
\end{aligned}$$

We thus have

$$\int \sin(\ln(x))\,dx = x\sin(\ln(x)) - x\cos(\ln(x)) - \int \sin(\ln(x))\,dx\,,$$

from which it follows that

$$2\int \sin(\ln(x))\,dx = x\sin(\ln(x)) - x\cos(\ln(x))\,,$$

so

$$\int \sin(\ln(x))\,dx = \frac{1}{2}\left[x\sin(\ln(x)) - x\cos(\ln(x))\right] + C\,,$$

remembering at the last minute that we are dealing with an indefinite integral, so a generic constant of integration should appear in the final answer. □

* All else failing, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`

Aɴ Aʟᴛᴇʀɴᴀᴛᴇ Aᴘᴘʀᴏᴀᴄʜ. Question **1** can also be done, or at least begun, by taking inspiration from trigonometric substitutions, where one substitutes a function in a new variable for x. In this case, one can try the substitution $x = e^t$, so $dx = e^t\,dt$, to get rid of the natural logarithm:

$$\int \sin(\ln(x))\,dx = \int \sin\left(\ln\left(e^t\right)\right) \cdot e^t\,dt = \int e^t \sin(t)\,dt$$

After doing so, one may continue with integration by parts without having to resort to the "dummy product" trick, though one will still have to use parts twice and solve for $\int e^t \sin(t)\,dt$, after which one will have to undo the substitution to put the antiderivative back in terms of x. ■

In questions **2** and **3**, and assuming $n \geq 0$, let $P_n(x)$ denote the polynomial such that $\int x^n e^x\,dx = P_n(x)e^x + C$. Note that $P_0(x) = 1$ because $x^0 = 1$ and e^x is its own antiderivative.

2. Show that $P_n(x) = x^n - P_{n-1}(x)$ for $n \geq 1$. *[3]*

Nᴏᴛᴇ: As many noticed, this formula is actually incorrect because something is missing. The correct formula is $P_n(x) = x^n - nP_{n-1}(x)$. The point to leaving out the n was to try to get those attempting the problem to realize that one cannot always trust what one is given. It was done on purpose this time, but textbooks, reference works, and computer software are created and ultimately checked by humans, who make honest mistakes ...

Sᴏʟᴜᴛɪᴏɴ. Assume $n \geq 1$. On the one hand, we have that $\int x^n e^x\,dx = P_n(x)e^x + C$ from the definition of $P_n(x)$. On the other hand, we can apply integration by parts to this integral, with $u = x^n$ and $v' = e^x$, so $u' = nx^{n-1}$ and $v = e^x$, so

$$\int x^n e^x\,dx = x^n e^x - \int nx^{n-1}e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx\,.$$

By the definition of $P_{n-1}(x)$, it follows that

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx = x^n e^x - P_{n-1}(x)e^x + C = \left(x^n - P_{n-1}(x)\right)e^x + C\,,$$

and hence that

$$P_n(x)e^x + C = \int x^n e^x\,dx = \left(x^n - P_{n-1}(x)\right)e^x + C\,.$$

Comparing the left and right ends of this sequence of equations, this is only possible if $P_n(x) = x^n - nP_{n-1}(x)$, as desired. ■

3. Use the relation in **2** to find $P_n(x)$ for $n = 1$, 2, 3, and 4. *[2]*

SOLUTION. As noted on this assignment, it follows from $\int x^0 e^x \, dx = \int e^x \, dx = e^x + C$ that $P_0(x) = 1$. Applying the (correct!) formula from **2**, namely $P_n(x) = x^n - nP_{n-1}(x)$, over and over, it then follows that:

$$\begin{aligned}
P_1(x) &= x^1 - 1P_0(x) = x - 1 \cdot 1 = x - 1 \\
P_2(x) &= x^2 - 2P_1(x) = x^2 - 2(x - 1) = x^2 - 2x + 2 \\
P_3(x) &= x^3 - 3P_2(x) = x^3 - 3\left(x^2 - 2x + 2\right) = x^3 - 3x^2 + 6x - 6 \\
P_4(x) &= x^4 - 4P_3(x) = x^4 - 4\left(x^3 - 3x^2 + 6x - 6\right) = x^4 - 4x^3 + 12x^2 - 24x + 24
\end{aligned}$$

This kind of trick was valuable in the days before computers, because it let one find the antiderivative of $\int x^n e^x \, dx$ for large n with a lot less computation than integrating it directly. ■