Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2022

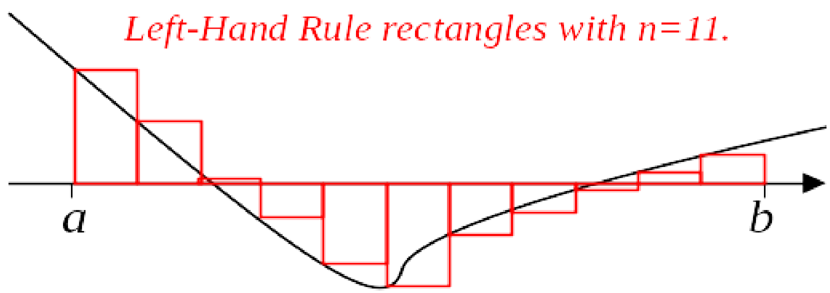
Solutions to Assignment #1
Computing definite integrals with the Left-Hand Rule

Recall that the definite integral $\int_a^b f(x)\,dx$ is the signed or weighted area of the region between $y = f(x)$ and the x-axis for $a \leq x \leq b$, where area above the x-axis is added and area below the x-axis is subtracted. It seems to be pretty hard to turn this idea into a complete and precise definition that can be used to prove all the basic properties of the definite integral, much less prove the Fundamental Theorem of Calculus, which relates the definite integral to computing antiderivatives. Indeed, many first-year calculus textbooks give highly simplified versions or even skip it entirely.[†] The definition given in §7.2 of our textbook is one of the more common highly simplified versions of this definition, often called the Left-Hand Rule. Please (at least!) skim through §7.2 before doing this assignment; for your convenience a summary of what you will need to know follows:

Left-Hand Rule. Suppose $f(x)$ is defined for all x in $[a, b]$ and is continuous at all but finitely many points of $[a, b]$. Then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{b-a}{n} f\left(a + (i-1)\cdot\frac{b-a}{n}\right)\right]$$

The idea is to divide up the interval $[a, b]$ into n subintervals of equal width $\triangle x = \frac{b-a}{n}$, so the ith subinterval, going from left to right and where $1 \leq i \leq n$, will be $\left[(i-1)\cdot\frac{b-a}{n}, i\cdot\frac{b-a}{n}\right]$. Each subinterval serves as the base of a rectangle of height $h_i = f\left(a + (i-1)\cdot\frac{b-a}{n}\right)$, which must then have area $\triangle x \cdot h_i = \frac{b-a}{n} f\left(a + (i-1)\cdot\frac{b-a}{n}\right)$.



It's called the Left-Hand Rule because it uses the left endpoint of each subinterval to evaluate $f(x)$ at to determine the height of the rectangle which has that subinterval as

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[†] You can find the simplest version of a precise and complete definition of the definite integral that your instructor knows of in *A Precise Definition of the Definite Integral*, a handout which is among the supplementary materials in the *Course Content* section of the course Blackboard site.

a base. Note that when the function dips below the x-axis, the rectangles have negative height and so the area formula gives negative areas.

The sum of the areas of these rectangles, the *nth Left-Hand Rule sum for* $\int_a^b f(x)\,dx$, namely $\displaystyle\sum_{i=1}^{n} \frac{b-a}{n} f\left(a + (i-1)\cdot\frac{b-a}{n}\right)$, approximates the area computed by $\int_a^b f(x)\,dx$. As we increase n and so shrink the width of the rectangles we get better and better approximations to the definite integral. The Left-Hand Rule will, in principle, properly compute $\int_a^b f(x)\,dx$ as long as $f(x)$ has at most finitely many removable or jump discontinuities and no vertical asymptotes in the interval $[a,b]$. Even some basic properties of definite integrals are hard to get if one were to try to use the Left-Hand Rule as the definition:

1. Suppose $f(x)$ is a function which is defined and continuous – and hence is integrable – on $\left[-1,\sqrt{2}\right]$. Explain why we would have a problem justifying

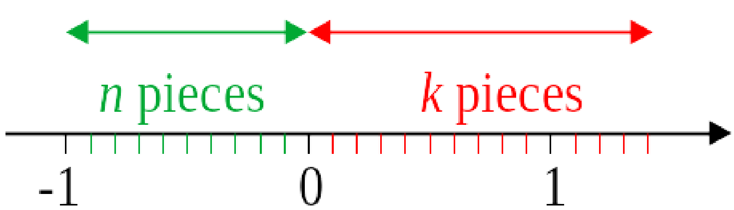
$$\int_{-1}^{0} f(x)\,dx + \int_{0}^{\sqrt{2}} f(x)\,dx = \int_{-1}^{\sqrt{2}} f(x)\,dx$$

if we used the Left-Hand Rule (or any rule that relies on subdividing $[a,b]$ into equal subintervals) as the actual definition of $\int_a^b f(x)\,dx$. *[2]*

Hint. It matters here that $\sqrt{2}$ is irrational.

Solution. The easiest way to get the integrals to add up properly if one uses the Left-Hand Rule as the definition would be if one could combine the Left-Hand Rule limits on the left-hand side to make the limit on the right-hand side. This, because a sum for the entire interval $\left[-1,\sqrt{2}\right]$ must have rectangles of equal width, is only feasible if the two sums have rectangles that have the same width, which means that one must be able to subdivide the intervals $[-1,0]$ and $\left[0,\sqrt{2}\right]$ into finitely many pieces of equal width to those in the other interval.

Suppose, by way of contradiction, that it were actually possible to divide $[-1,0]$ into n equal pieces, each of which would have to have width $\dfrac{0-(-1)}{n} - \dfrac{1}{n}$, and at the same time divide up $\left[0,\sqrt{2}\right]$ into k equal pieces, each of which would have to have width $\dfrac{\sqrt{2}-0}{k} = \dfrac{\sqrt{2}}{k}$, such that the pieces in each interval are equal in width to those in the other interval, *i.e.* $\dfrac{1}{n} = \dfrac{\sqrt{2}}{k}$.



However, if $\frac{1}{n} = \frac{\sqrt{2}}{k}$ were true, then we would have that $\sqrt{2} = \frac{k}{n}$, where n and k are positive integers. This would mean that $\sqrt{2}$ was a ratio of integers, that is, that it was a rational number. Since $\sqrt{2}$ is famously irrational, it cannot be written as a ratio of integers, and so it is impossible to subdivide the two intervals into equal pieces of the same width as those in the other interval.

It follows that if one uses the Left-Hand Rule as the definition of the definite integral, the easiest way to show the given additive property of definite integrals is not going to work. One can try to work around this (generally requiring some very nasty limit arguments), but serious definitions of the definite integral use variable-width subintervals to get their rectangles. ■

2. Compute the nth Left-Hand Rule sum for $\int_0^4 \left(x^2 + 2x + 3\right)\,dx$ for $n = 4$, 8, 16, and 32. *[2]*

Hint. Use mathematical software such as `SageMath`. (Unless you're a mathochist. :-)

SOLUTION. The nth Left-Hand Rule sum for $\int_0^4 \left(x^2 + 2x + 3\right)\,dx$ is given by

$$\sum_{i=1}^{n} \frac{4-0}{n} f\left(0 + (i-1)\cdot\frac{4-0}{n}\right) = \sum_{i=1}^{n} \frac{4}{n} f\left((i-1)\cdot\frac{4}{n}\right),$$

where $f(x) = x^2 + 2x + 3$. One could rearrange this in various ways to make it easier to compute, such as factoring the rectangle width $\frac{4}{n}$ out of the sum, but since we're getting a computer to do the work, and the job will not strain it in the least, why bother?

Here is your instructor's interaction with `SageMath` to do this problem. Note that this is using a local installation of `SageMath` in a terminal window rather than using a remote installation, such as `sage.trentu.ca`, via a Jupyter notebook.

```
sage:  f(x) = x^2 + 2*x + 3
sage:  var("n")
n
sage:  var("i")
i
sage:  g(n) = sum((4/n)*f((i-1)*4/n),i,1,n)
sage:  g(4)
38
sage:  g(8)
87/2
sage:  g(16)
371/8
sage:  g(32)
1531/32
```

```
sage:  N(87/2)
43.5000000000000
sage:  N(371/8)
46.3750000000000
sage:  N(1531/32)
47.8437500000000
```

The last few commands are to check on what the fractions really amount to. Note that the values are getting closer to the true value of $\frac{148}{3} \approx 49.3333$, as computed in the next two solutions. ■

3. Use the Left-Hand Rule to compute $\displaystyle\int_0^4 \left(x^2+2x+3\right)\,dx$ precisely. *[4]*

Hint. You'll need to do some algebra before taking the limit and may use the summation formulas $\displaystyle\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ and $\displaystyle\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

Solution. If $\displaystyle\int_a^b f(x)\,dx = \int_0^4 \left(x^2+2x+3\right)\,dx$, then $a=0$, $b=4$, and $f(x)=x^2+2x+3$ when we apply the Left-Hand Rule definition of the definite integral. We plug these into the Left-Hand Rule formula and simplify:

$$
\begin{aligned}
\int_0^4 \left(x^2+2x+3\right)\,dx &= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{4-0}{n} f\left(0+(i-1)\cdot\frac{4-0}{n}\right)\right] \\
&= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{4}{n} f\left((i-1)\cdot\frac{4}{n}\right)\right] = \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n} f\left(\frac{4}{n}(i-1)\right) \\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left[\left(\frac{4}{n}(i-1)\right)^2 + 2\left(\frac{4}{n}(i-1)\right)+3\right] \\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left[\frac{16}{n^2}\left(i^2-2i+1\right)+\frac{8}{n}\left(i-1\right)\right)+3\right] \\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left[\frac{16}{n^2}i^2-\frac{32}{n^2}i+\frac{16}{n^2}+\frac{8}{n}i-\frac{8}{n}+3\right]
\end{aligned}
$$

We now divide up the sum we have into smaller, simpler, sums, isolate the the sums we have summation formulas for – namely $\displaystyle\sum_{i=1}^{n} 1 = n$, $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, and $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ – replace them with these summation formulas, and then simplify away until we can compute the limit:

$$
\begin{aligned}
\int_0^4 \left(x^2+2x+3\right)\,dx &= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left[\frac{16}{n^2}i^2 - \frac{32}{n^2}i + \frac{16}{n^2} + \frac{8}{n}i - \frac{8}{n} + 3\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \left[\left(\sum_{i=1}^{n} \frac{16}{n^2}i^2\right) - \left(\sum_{i=1}^{n} \frac{32}{n^2}i\right) + \left(\sum_{i=1}^{n} \frac{16}{n^2}\right)\right. \\
&\qquad \left. + \left(\sum_{i=1}^{n} \frac{8}{n}i\right) - \left(\sum_{i=1}^{n} \frac{8}{n}\right) + \left(\sum_{i=1}^{n} 3\right)\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \left[\frac{16}{n^2}\left(\sum_{i=1}^{n} i^2\right) - \frac{32}{n^2}\left(\sum_{i=1}^{n} i\right) + \frac{16}{n^2}\left(\sum_{i=1}^{n} 1\right)\right. \\
&\qquad \left. + \frac{8}{n}\left(\sum_{i=1}^{n} i\right) - \frac{8}{n}\left(\sum_{i=1}^{n} 1\right) + 3\left(\sum_{i=1}^{n} 1\right)\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \left[\frac{16}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{32}{n^2} \cdot \frac{n(n+1)}{2} + \frac{16}{n^2} \cdot n\right. \\
&\qquad \left. + \frac{8}{n} \cdot \frac{n(n+1)}{2} - \frac{8}{n} \cdot n + 3 \cdot n\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \left[\frac{8}{3} \cdot \frac{(n+1)(2n+1)}{n} - 16 \cdot \frac{n+1}{n} + \frac{16}{n}\right. \\
&\qquad \left. + 4(n+1) - 8 + 3n\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \left[\frac{8}{3} \cdot \frac{2n^2+3n+1}{n} - 16 \cdot \left(1 + \frac{1}{n}\right) + \frac{16}{n}\right. \\
&\qquad \left. + 4n + 4 - 8 + 3n\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \left[\frac{16n}{3} + 8 + \frac{8}{3n} - 16 - \frac{16}{n} + \frac{16}{n} + 7n - 4\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \left[\frac{37n}{3} + \frac{8}{3n} - 12\right] = \lim_{n\to\infty} \left[\frac{148}{3} + \frac{32}{3n^2} - \frac{48}{n}\right] \\
&= \frac{148}{3} + 0 + 0 = \frac{148}{3} \approx 49.3333 \qquad \blacksquare
\end{aligned}
$$

Note. The above calculation should be a clue as to why using the Left-Hand Rule, or similar techniques, is not practical for computing definite integrals precisely.

4. Compute $\displaystyle\int_0^4 \left(x^2+2x+3\right)\,dx$ precisely using antiderivatives. *[2]*

SOLUTION. *[Finally, something straightforward and easy! :-)]* The Power Rule and the linearity of the definite integral are our friends here:

$$\int_0^4 \left(x^2+2x+3\right)\,dx = \left(\frac{x^3}{3}+2\frac{x^2}{2}+3x\right)\Bigg|_0^4 = \left(\frac{4^3}{3}+4^2+3\cdot 4\right) - \left(\frac{0^3}{3}+0^2+3\cdot 0\right)$$
$$= \left(\frac{64}{3}+16+12\right) - 0 = \frac{64}{3}+28 = \frac{64}{3}+\frac{84}{3} = \frac{148}{3} \approx 49.3333 \quad \blacksquare$$