Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2022
Solutions to the Final Examination
11:00-14:00 on Saturday, 23 April, in Wenjack.

Time: *3 hours.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts **X**, **Y**, and **Z**, and, if you wish, part **W**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Open book, most any calculator, one head-mounted neural net.

Part X. Do all four (4) of **1**–**4**.

1. Evaluate any *four* (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

a. $\displaystyle\int_{12}^{14} (x-13)^6\,dx$ **b.** $\displaystyle\int \frac{1}{z^2+3z+2}\,dz$ **c.** $\displaystyle\int_0^1 \frac{y\arctan(y)}{y+y^3}\,dy$

d. $\displaystyle\int u^3\sin\left(u^2\right)\,du$ **e.** $\displaystyle\int_0^\infty \frac{1}{(2v+3)^2}\,dv$ **f.** $\displaystyle\int \frac{2}{\sqrt{1+4w^2}}\,dw$

Solutions. **a.** We will use the substitution $w = x-13$, so $dw = dx$, and change the limits as we go along: $\begin{matrix} x & 12 & 14 \\ w & -1 & 1 \end{matrix}$

$$\int_{12}^{14} (x-13)^6\,dx = \int_{-1}^{1} w^6\,dw = \left.\frac{w^7}{7}\right|_{-1}^{1} = \frac{1^7}{7} - \frac{(-1)^7}{7} = \frac{1}{7} - \frac{-1}{7} = \frac{2}{7} \quad \square$$

b. *Using partial fractions.* Observe that $z^2+3z+2 = (z+2)(z+1)$; it follows that

$$\frac{1}{z^2+3z+2} = \frac{1}{(z+2)(z+1)} = \frac{A}{z+2} + \frac{B}{z+1} = \frac{A(z+1)+B(z+2)}{(z+2)(z+1)} = \frac{(A+B)z+(A+2B)}{(z+2)(z+1)},$$

where A and B are as yet unknown constants. Comparing the numerators at the beginning and the end above, we see that $A+B=0$ and $A+2B=1$. Thus $B = (A+2B)-(A+B) = 1-0 = 1$, and so $0 = A+B = A+1$, which yields $A=-1$. Thus $\dfrac{1}{z^2+3z+2} = \dfrac{-1}{z+2} + \dfrac{1}{z+1}$. We will finish the job with the help of the substitutions $u = z+2$ and $w = z+1$, so $du = dw = dz$.

$$\begin{aligned}\int \frac{1}{z^2+3z+2}\,dz &= \int \frac{1}{z^2+3z+2}\,dz = \int \left[\frac{-1}{z+2} + \frac{1}{z+1}\right]\,dz = \int \left[\frac{1}{z+1} - \frac{1}{z+2}\right]\,dz \\ &= \int \frac{1}{z+1}\,dz - \int \frac{-1}{z+2}\,dz = \int \frac{1}{w}\,dw - \int \frac{1}{u}\,du \\ &= \ln(w) - \ln(u) + C = \ln(z+1) - \ln(z+2) + C = \ln\left(\frac{z+1}{z+2}\right) + C \quad \square\end{aligned}$$

b. *Using a trigonometric substitution.* Completing the square, we get that $z^2 + 3z + 2 = \left(z + \frac{3}{2}\right) - \left(\frac{3}{2}\right)^2 + 2 = \left(z + \frac{3}{2}\right)^2 - \frac{1}{4}$. We will first simplify matters using the substitution $\frac{1}{2}u = z + \frac{3}{2}$, so $dz = \frac{1}{2}\,du$, as follows:

$$\int \frac{1}{z^2+3z+2}\,dz = \int \frac{1}{\left(z+\frac{3}{2}\right)^2 - \frac{1}{4}}\,dz == \int \frac{1}{\left(\frac{1}{2}u\right)^2 - \frac{1}{4}} \cdot \frac{1}{2}\,du = 2\int \frac{1}{u^2-1}\,du$$

We now apply the trigonometric substitution $u = \sec(\theta)$, so $du = \sec(\theta)\tan(\theta)\,d\theta$:

$$\begin{aligned}\int \frac{1}{z^2+3z+2}\,dz &= 2\int \frac{1}{u^2-1}\,du = 2\int \frac{1}{\sec^2(\theta)-1} \cdot \sec(\theta)\tan(\theta)\,d\theta \\ &= 2\int \frac{\sec(\theta)\tan(\theta)}{\tan^2(\theta)}\,d\theta = 2\int \frac{\sec(\theta)}{\tan(\theta)}\,d\theta = 2\int \frac{\frac{1}{\cos(\theta)}}{\frac{\sin(\theta)}{\cos(\theta)}}\,d\theta \\ &= 2\int \frac{1}{\sin(\theta)}\,d\theta = 2\int \csc(\theta)\,d\theta \quad \textit{[Consults integral table.]} \\ &= 2\ln\left(\csc(\theta) - \cot(\theta)\right) + C\end{aligned}$$

All that remains is to substitute back. Let's do the algebra for that outside the integral:

$$\begin{aligned}u = \sec(\theta) &\implies u = \frac{1}{\cos(\theta)} \implies \cos(\theta) = \frac{1}{u} \\ &\implies \sin(\theta) = \sqrt{1-\cos^2(\theta)} = \sqrt{1 - \frac{1}{u^2}} = \frac{\sqrt{u^2-1}}{u} \\ &\implies \csc(\theta) = \frac{1}{\sin(\theta)} = \frac{u}{\sqrt{u^2-1}} \quad \& \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{\frac{1}{u}}{\frac{\sqrt{u^2-1}}{u}} = \frac{1}{\sqrt{u^2-1}}\end{aligned}$$

It follows that:

$$\csc(\theta) - \cot(\theta) = \frac{u}{\sqrt{u^2-1}} - \frac{1}{\sqrt{u^2-1}} = \frac{u-1}{\sqrt{(u-1)(u+1)}} = \sqrt{\frac{u-1}{u+1}}$$

Since $\frac{1}{2}u = z + \frac{3}{2}$, we have $u = 2z+3$, so $u - 1 = 2z + 2$ and $u + 1 = 2z + 4$. Then

$$\csc(\theta) - \cot(\theta) = \sqrt{\frac{u-1}{u+1}} = \sqrt{\frac{2z+2}{2z+4}} = \sqrt{\frac{z+1}{z+2}},$$

and so

$$\begin{aligned}\int \frac{1}{z^2+3z+2}\,dz &= 2\ln\left(\csc(\theta)-\cot(\theta)\right) + C = 2\ln\left(\sqrt{\frac{z+1}{z+2}}\right) + C \\ &= \ln\left(\left(\sqrt{\frac{z+1}{z+2}}\right)^2\right) + C = \ln\left(\frac{z+1}{z+2}\right) + C \quad \square\end{aligned}$$

c. A little cancellation will simplify matters, followed by the substitution $u = \arctan(y)$, so $du = \dfrac{1}{1+y^2}\,dy$, changing the limits as we go along: $\begin{matrix} y & 0 & 1 \\ u & 0 & \pi/4 \end{matrix}$

$$\int_0^1 \frac{y\arctan(y)}{y+y^3}\,dy = \int_0^1 \frac{y\arctan(y)}{y\left(1+y^2\right)}\,dy = \int_0^1 \frac{\arctan(y)}{1+y^2}\,dy = \int_0^{\pi/4} u\,du$$

$$= \left.\frac{u^2}{2}\right|_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4}\right)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}\cdot\frac{\pi^2}{16} - 0 = \frac{\pi^2}{32} \quad \square$$

d. *Substitution, then integration by parts.* The u^2 in $\sin\left(u^2\right)$ is a clue as to what to try first. We will use the substitution $w = u^2$, so $dw = 2u\,du$ and $u\,du = \frac{1}{2}\,dw$, to simplify the integrand, and then use integration by parts, with $s = w$ and $t' = \sin(w)$, so $s' = 1$ and $t = -\cos(w)$.

$$\begin{aligned} \int u^3 \sin\left(u^2\right)\,du &= \int u^2 \sin\left(u^2\right)\,u\,du = \int w\sin(w)\,\frac{1}{2}\,dw \\ &= \frac{1}{2}\left[w\left(-\cos(w)\right) - \int 1\left(-\cos(w)\right)\,dw\right] \\ &= \frac{1}{2}\left[-w\cos(w) + \int \cos(w)\,dw\right] \\ &= \frac{1}{2}\left[-w\cos(w) + \sin(w)\right] + C \\ &= -\frac{1}{2}u^2\cos\left(u^2\right) + \frac{1}{2}\sin\left(u^2\right) + C \quad \square \end{aligned}$$

d. *Integration by parts, then substitution.* The problem with doing parts first is deciding how to divide up the integrand. Most ways of doing so leave a mess or just don't work, but letting $s = u^2$ and $t' = u\sin\left(u^2\right)$ isn't too bad. We get $s' = 2u$ and

$$t = \int u\sin\left(u^2\right)\,du \qquad \text{Substitute } w = u^2\text{, so } dw = 2u,\ du \text{ and } u\,du = \tfrac{1}{2}\,dw.$$

$$= \int \sin(w)\,\frac{1}{2}dw = -\frac{1}{2}cos(w) = -\frac{1}{2}\cos\left(u^2\right).$$

We will use the same substitution that we did in working out t above in the main computation:

$$\begin{aligned} \int u^3\sin\left(u^2\right)\,du &= u^2\left(-\frac{1}{2}\cos\left(u^2\right)\right) - \int 2u\left(-\frac{1}{2}\cos\left(u^2\right)\right)\,du \\ &= -\frac{1}{2}u^2\cos\left(u^2\right) + \int u\cos\left(u^2\right)\,du \\ &= -\frac{1}{2}u^2\cos\left(u^2\right) + \int \cos(w)\,\frac{1}{2}\,dw \\ &= -\frac{1}{2}u^2\cos\left(u^2\right) + \frac{1}{2}\sin(w) + C \\ &= -\frac{1}{2}u^2\cos\left(u^2\right) + \frac{1}{2}\sin\left(u^2\right) + C \quad \square \end{aligned}$$

e. We will use the substitution $w = 2v + 3$, so $dw = 2dv$ and $dv = \frac{1}{2}\,dw$, and change the limits as we go along: $\begin{matrix} v & 0 & \infty \\ w & 3 & \infty \end{matrix}$.

$$\begin{aligned} \int_0^\infty \frac{1}{(2v+3)^2}\,dv &= \int_3^\infty \frac{1}{w^2} \cdot \frac{1}{2}\,dw = \lim_{a\to\infty} \frac{1}{2} \int_3^a w^{-2}\,dw = \lim_{a\to\infty} \frac{1}{2} \cdot \left.\frac{w^{-1}}{-1}\right|_3^\infty \\ &= \lim_{a\to\infty} \left.\frac{-1}{2w}\right|_3^\infty = \lim_{a\to\infty} \left(\frac{-1}{2a} - \frac{-1}{2\cdot 3}\right) = \lim_{a\to\infty} \left(\frac{-1}{2a} + \frac{1}{6}\right) \\ &= 0 + \frac{1}{6} = \frac{1}{6} \quad \square \end{aligned}$$

f. We will use the trigonometric substitution $w = \frac{1}{2}\tan(\theta)$, so $dw = \frac{1}{2}\sec^2(\theta)\,d\theta$. Note that $\tan(\theta) = 2w$ and $\sec(\theta) = \sqrt{\sec^2(\theta)} = \sqrt{1+\tan^2(\theta)} = \sqrt{1+(2w)^2} = \sqrt{1+4w^2}$.

$$\begin{aligned} \int \frac{2}{\sqrt{1+4w^2}}\,dw &= \int \frac{2}{\sec(\theta)} \cdot \frac{1}{2}\sec^2(\theta)\,d\theta = \int \sec(\theta)\,d\theta \\ &= \ln\left(\tan(\theta)+\sec(\theta)\right) + C = \ln\left(2w + \sqrt{1+4w^2}\right) + C \quad \blacksquare \end{aligned}$$

2. Determine whether the series converges in any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

$$\textbf{a. } \sum_{n=0}^\infty 2^{-n^2} \qquad \textbf{b. } \sum_{m=1}^\infty \frac{1}{\cos(m\pi)\cdot\sqrt{m}} \qquad \textbf{c. } \sum_{i=0} \frac{i}{3^i}$$

$$\textbf{d. } \sum_{j=1}^\infty \frac{3^j}{j} \qquad \textbf{e. } \sum_{k=1}^\infty \frac{k!}{(k-1)!\cdot k^2} \qquad \textbf{f. } \sum_{a=0}^\infty \frac{\sqrt{a}}{1+a^2}$$

SOLUTION. **a.** *Root Test.* Here goes:

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|2^{-n^2}\right|^{1/n} = \lim_{n\to\infty} 2^{-n^2\cdot(1/n)} = \lim_{n\to\infty} 2^{-n} = \lim_{n\to\infty} \frac{1}{2^n} \begin{matrix}\to 1\\ \to \infty\end{matrix} = 0$$

Since $0 < 1$, it follows by the Root Test that the series converges absolutely. $\square$

a. *Ratio Test.* Here goes:

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{2^{-(n+1)^2}}{2^{-n^2}}\right| = \lim_{n\to\infty} \frac{2^{-n^2-2n-1}}{2^{-n^2}} = \lim_{n\to\infty} \frac{2^{-n^2}2^{-2n-1}}{2^{-n^2}} \\ &= \lim_{n\to\infty} 2^{-2n-1} = \lim_{n\to\infty} \frac{1}{2^{2n+1}} \begin{matrix}\to 1\\ \to \infty\end{matrix} = 0 \end{aligned}$$

Since $0 < 1$, it follows by the Ratio Test that the series converges absolutely. $\square$

a. *Basic Comparison Test.* The geometric series $\sum_{n=0}^\infty 2^{-n} = \sum_{n=0}^\infty \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ converges because it has a common ratio of $r = \frac{1}{2}$ and so $|r| = \frac{1}{2} < 1$. Since $n^2 \geq n$ for all

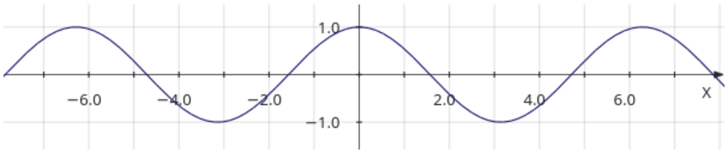
$n \geq 0$, we have $0 < 2^{-n^2} = \dfrac{1}{2^{n^2}} < \dfrac{1}{2^n} = 2^{-n}$. It follows by the (Basic) Comparison Test that $\displaystyle\sum_{n=0}^{\infty} 2^{-n^2}$ converges as well. □

a. *Limit Comparison Test.* The geometric series $\displaystyle\sum_{n=0}^{\infty} 2^{-n} = \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ converges because it has a common ratio of $r = \frac{1}{2}$, so $|r| = \frac{1}{2} < 1$. Since both series are made up of positive terms and

$$\lim_{n\to\infty} \frac{2^{-n^2}}{2^{-n}} = \lim_{n\to\infty} \frac{1/2^{n^2}}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^{n^2}} = \lim_{n\to\infty} 2^{n-n^2} = \lim_{n\to\infty} 2^{n(1-n)} = 0$$

(because $n(1-n) \to -\infty$ as $n \to \infty$), it follows by the Limit Comparison Test that $\displaystyle\sum_{n=0}^{\infty} 2^{-n^2}$ converges as well. □

b. *Alternating Series Test.* The key to this one is that $\cos(m\pi) = (-1)^m$, which is easy to check if you know what the graph of $y = \cos(x)$ looks like:



We will apply the Alternating Series Test:

First, $\dfrac{1}{\cos(m\pi)\cdot\sqrt{m}} = \dfrac{(-1)^m}{\cos(m\pi)}$ alternates sign as m runs through the positive integers, since $\dfrac{1}{\sqrt{m}} > 0$ for all $m \geq 1$.

Second, since $\sqrt{x}$ is an increasing function, we have $\sqrt{m+1} > \sqrt{m}$, so it follows that $\left|\frac{1}{\cos((m+1)\pi)\cdot\sqrt{m+1}}\right| = \left|\frac{(-1)^{m+1}}{\sqrt{m+1}}\right| = \frac{1}{\sqrt{m+1}} < \frac{1}{\sqrt{m}} = \left|\frac{(-1)^m}{\sqrt{m}}\right| = \left|\frac{1}{\cos(m\pi)\cdot\sqrt{m}}\right|$ for all $m \geq 1$.

Finally, $\displaystyle\lim_{m\to\infty} \left|\frac{1}{\cos(m\pi)\cdot\sqrt{m}}\right| = \lim_{m\to\infty} \frac{1}{\sqrt{m}} \begin{matrix}\to 1\\ \to \infty\end{matrix} = 0.$

It now follows by the Alternating Series Test that the series $\displaystyle\sum_{m=1}^{\infty} \frac{1}{\cos(m\pi)\cdot\sqrt{m}}$ converges.

For those who care, the series converges conditionally because the corresponding series of positive terms, $\displaystyle\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} = \sum_{m=1}^{\infty} \frac{1}{m^{1/2}}$, diverges by the p-Test as it has $p = \frac{1}{2} < 1$. ■

c. *Ratio Test.* Here we go:

$$\lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| = \lim_{i\to\infty} \left|\frac{\frac{i+1}{3^{i+1}}}{\frac{i}{3^i}}\right| = \lim_{i\to\infty} \frac{i+1}{3^{i+1}} \cdot \frac{3^i}{i} = \lim_{i\to\infty} \frac{1}{3}\left(1 + \frac{1}{i}\right) = \frac{1}{3}(1+0) = \frac{1}{3}$$

Since $\frac{1}{3} < 1$, it follows by the Ratio Test that the series $\displaystyle\sum_{i=0}^{\infty} \frac{i}{3^i}$ converges absolutely. $\square$

c. *Integral Test.* We will work out the corresponding improper integral, with a little help from the technique of integration by parts and a side of l'Hôpital's Rule. Note that the terms of the given series come from pluggin integer values into the function $f(x) = \frac{x}{3^x}$, which is positive for $x > 0$ and decreasing for $x > \frac{1}{\ln(3)}$. (Why?)

$$\begin{aligned}
\int_0^{\infty} \frac{x}{3^x}\,dx &= \lim_{a\to\infty} \int_0^a x3^{-x}\,dx \qquad \begin{array}{ll} u = x & v' = 3^{-x} \\ u' = 1 & v = \dfrac{-3^{-x}}{\ln(3)} \end{array} \\
&= \lim_{a\to\infty} \left[x \cdot \frac{-3^{-x}}{\ln(3)} \Big|_0^a - \int_0^a 1 \cdot \frac{-3^{-x}}{\ln(3)}\,dx \right] \\
&= \lim_{a\to\infty} \left[a \cdot \frac{-3^{-a}}{\ln(3)} - 0 \cdot \frac{-3^{-0}}{\ln(3)} + \frac{-3^{-x}}{(\ln(3))^2} \Big|_0^a \right] \\
&= \lim_{a\to\infty} \left[-\frac{a}{3^a \ln(a)} - 0 + \frac{-3^{-a}}{(\ln(3))^2} - \frac{-3^{-0}}{(\ln(3))^2} \right] \\
&= \lim_{a\to\infty} \left[\frac{1}{(\ln(3))^2} - \frac{a}{3^a \ln(a)} - \frac{1}{3^a (\ln(3))^2} \right] \\
&= \frac{1}{(\ln(3))^2} - \left[\lim_{a\to\infty} \frac{a}{3^a \ln(a)} \begin{array}{l} \to \infty \\ \to \infty \end{array} \right] - \left[\lim_{a\to\infty} \frac{1}{3^a (\ln(3))^2} \begin{array}{l} \to \ln(3) \\ \to \infty \end{array} \right] \\
&= \frac{1}{(\ln(3))^2} - \left[\lim_{a\to\infty} \frac{\frac{d}{da} a}{\ln(3) \frac{d}{da} 3^a \ln(a)} \right] - 0 \\
&= \frac{1}{(\ln(3))^2} - \lim_{a\to\infty} \frac{1}{3^a (\ln(3))^2} \begin{array}{l} \to 1 \\ \to \infty \end{array} = \frac{1}{(\ln(3))^2} - 0 = \frac{1}{(\ln(3))^2}
\end{aligned}$$

Since the corresponding improper integral converges, the Integral Test tells us that the series $\displaystyle\sum_{i=0}^{\infty} \frac{i}{3^i}$ converges as well. $\square$

c. *Basic Comparison Test.* We will compare the given series, $\displaystyle\sum_{i=0}^{\infty} \frac{i}{3^i}$, to the geometric series $\displaystyle\sum_{i=0}^{\infty} \frac{1}{3^{i/2}} = \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^i$, which converges because it has a common ratio of $\frac{1}{\sqrt{3}} \approx 0.5774$ that has an absolute value less than 1.

The key to showing that $\frac{i}{3^i} < \frac{1}{3^{i/2}} = \frac{3^{i/2}}{3^i}$ for all $i \geq 0$ is showing that $i < 3^{i/2}$ for all $i \geq 0$. Observe that $0 < 3^{0/2} = 1$, $1 < 3^{1/2} = \sqrt{3} \approx 1.7321$, and $2 < 3^{2/2} = 3$. Since

$\frac{d}{dx}x = 1$ and $\frac{d}{dx}3^{x/2} = \ln(3)3^{x/2}\frac{d}{dx}\left(\frac{x}{2}\right) = \ln(3)3^{x/2}\frac{1}{2} \geq \ln(3)3^{2/2}\frac{1}{2} = \frac{3}{2}\ln(3) \approx 1.6479 > 1$ for all $x \geq 2$, it follows that $3^{i/2}$ grows faster than i for $i \geq 2$. Since we already have $2 < 3^{2/2} = 3$, it follows that $i < 3^{i/2}$ for all $i > 2$ as well. Thus $0 \leq \frac{i}{3^i} < \frac{1}{3^{i/2}}$ for all $i \geq 0$.

Since we know that $\sum_{i=0}^{\infty} \frac{1}{3^{i/2}}$ converges, it now follows by the Basic Comparison Test that the given series, $\sum_{i=0}^{\infty} \frac{i}{3^i}$, converges as well. □

c. *Limit Comparison Test.* As with the use of the Basic Comparison Test above, we will compare the given series, $\sum_{i=0}^{\infty} \frac{i}{3^i}$, to the geometric series $\sum_{i=0}^{\infty} \frac{1}{3^{i/2}} = \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^i$. Again, the latter series converges because it has a common ratio of $\frac{1}{\sqrt{3}} \approx 0.5774$ that has an absolute value less than 1. We compute the limit required by the Limit Comparison Test and see what happens, with a little help from l'Hôpital's Rule:

$$\lim_{i\to\infty} \frac{\frac{i}{3^i}}{\frac{1}{3^{i/2}}} = \lim_{i\to\infty} \frac{i}{3^i} \cdot \frac{3^{i/2}}{1} = \lim_{i\to\infty} \frac{i}{3^{i/2}} = \lim_{x\to\infty} \frac{x}{3^{x/2}} \begin{matrix} \to \infty \\ \to \infty \end{matrix} = \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}3^{x/2}}$$
$$= \lim_{x\to\infty} \frac{1}{\ln(3)3^{x/2}\frac{d}{dx}\left(\frac{x}{2}\right)} = \lim_{x\to\infty} \frac{1}{\ln(3)3^{x/2}\frac{1}{2}} = \lim_{x\to\infty} \frac{2}{\ln(3)3^{x/2}} \begin{matrix} \to 2 \\ \to \infty \end{matrix} = 0$$

Since the limit is a real number, it follows that the given series, $\sum_{i=0}^{\infty} \frac{i}{3^i}$, converges because $\sum_{i=0}^{\infty} \frac{1}{3^{i/2}}$ does. □

d. *Ratio Test.* The series here is the reciprocal of the series in **c**, so the Ratio Test works in a very similar way, just with all of the fractions upside down and with the opposite conclusion. Here we go:

$$\lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| = \lim_{j\to\infty} \left|\frac{\frac{3^{j+1}}{j+1}}{\frac{3^j}{j}}\right| = \lim_{j\to\infty} \frac{3^{j+1}}{j+1} \cdot \frac{j}{3^j} = \lim_{j\to\infty} \frac{3j}{j+1} = 3 \lim_{j\to\infty} \frac{j}{j+1} \cdot \frac{\frac{1}{j}}{\frac{1}{j}}$$
$$= 3 \lim_{j\to\infty} \frac{1}{1+\frac{1}{j}} = 3 \cdot \frac{1}{1+0} = 3$$

Since $3 > 1$, it follows by the Ratio Test that the series $\sum_{j=1}^{\infty} \frac{3^j}{j}$ converges absolutely. □

NOTE: Since the series in **d** is the reciprocal of the series in **c**, we can also use the Basic and Limit Comparison Tests in a similar way. just with all the fractions upside down and with the opposite conclusion. If you're interested, give it a try, though, as in the solutions to **c** using the Comparison Tests, it's rather more work than using the Ratio Test. As with **c**, it is possible, at least in principle, to do **d** using the Integral Test. It's probably not a good idea to do **d** this way, though, since $\int \frac{3^x}{x}\,dx$ is much harder to integrate than $\int \frac{x}{3^x}\,dx$. On the other hand, there is one test that can be used for **d** that is not useful for **c**, and which is also a very efficient way to do **d**:

d. *Divergence Test.* Here we go, with a little help from l'Hôpital's Rule:

$$\begin{aligned}\lim_{j\to\infty} a_j &= \lim_{j\to\infty} \frac{3^j}{j} = \lim_{x\to\infty} \frac{3^x \to \infty}{x \to \infty} = \lim_{x\to\infty} \frac{\frac{d}{dx}3^x}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\ln(3)3^x}{1} \\ &= \ln(3) \lim_{x\to\infty} 3^x = \infty \neq 0\end{aligned}$$

Since the individual terms of the series do not go to 0 as $j \to \infty$, the Divergence Test tells us that the series must diverge. □

e. *Algebraic Simplification.* Observe that the given series

$$\sum_{k=1}^{\infty} \frac{k!}{(k-1)! \cdot k^2} = \sum_{k=1}^{\infty} \frac{(k-1)! \cdot k}{(k-1)! \cdot k \cdot k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

is really just the harmonic series in disguise. We know from class that the harmonic series diverges, or we could check using the p-Test (as $p = 1 - 0 = 1 \leq 1$) or the Integral Test (as $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty} \ln(x)|_1^a = \lim_{a\to\infty} (\ln(a) - \ln(0)) = \infty - 0 = \infty$). □

Note. In case you feel tempted to try it, the Ratio Test is inconclusive when applied to the series in **e**.

f. *Generalized p-Test.* $\sum_{a=0}^{\infty} \frac{\sqrt{a}}{1+a^2} = \sum_{a=0}^{\infty} \frac{a^{1/2}}{1+a^2}$ has $p = 2 - \frac{1}{2} = \frac{3}{2} > 1$, so the series converges by the Generalized p-Test. □

f. *Basic Comparison Test.* We will compare the given series to the series $\sum_{a=1}^{\infty} \frac{1}{a^{3/2}}$. Note that for all $a \geq 1$, $0 < \frac{\sqrt{a}}{1+a^2} < \frac{\sqrt{a}}{a^2} = \frac{a^{1/2}}{a^2} = \frac{1}{a^{3/2}}$. Since $\sum_{a=1}^{\infty} \frac{1}{a^{3/2}}$ converges by the p-Test, as it has $p = \frac{3}{2} > 1$, it follows by the Basic Comparison Test that $\sum_{a=1}^{\infty} \frac{\sqrt{a}}{1+a^2}$ converges as well. Hence so does the given series, $\sum_{a=0}^{\infty} \frac{\sqrt{a}}{1+a^2} = 0 + \sum_{a=1}^{\infty} \frac{\sqrt{a}}{1+a^2}$. □

f. *Limit Comparison Test.* We will compare the given series to the series $\sum_{a=1}^{\infty} \frac{1}{a^{3/2}}$.

$$\lim_{a\to\infty} \left| \frac{\frac{1}{a^{3/2}}}{\frac{\sqrt{a}}{1+a^2}} \right| = \lim_{a\to\infty} \frac{1}{a^{3/2}} \cdot \frac{1+a^2}{\sqrt{a}} = \lim_{a\to\infty} \frac{1+a^2}{a^2} = \lim_{a\to\infty} \left(\frac{1}{a^2} + 1 \right) = \frac{1}{0+1} = 1$$

Since the ratio of the terms from each series has a limit as $a \to \infty$, it follows by the Limit Comparison Test that the two series either both converge or both diverge. Since the series $\sum_{a=1}^{\infty} \frac{1}{a^{3/2}}$ converges by the p-Test because it has $p = \frac{3}{2} > 1$, the given series, $\sum_{a=0}^{\infty} \frac{\sqrt{a}}{1+a^2} = 0 + \sum_{a=1}^{\infty} \frac{\sqrt{a}}{1+a^2}$, must do so as well. ■

3. Do any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. Find the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{n}{17^n} x^n$.

b. Determine whether the series $\sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$ diverges, converges conditionally, or converges absolutely.

c. Find the volume of the solid obtained by revolving the region between $y = x - 4$ and $y = 0$, where $0 \le x \le 4$, about the x-axis.

d. Use the Left-Hand Rule to compute $\int_0^2 x\,dx$.

e. Find the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

f. Find the area of the finite region between $y = x$ and $y = x^3$.

Solution. **a.** As usual, our first resort is the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{n+1}{17^{n+1}} x^{n+1}}{\frac{n}{17^n} x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{17^{n+1}} \cdot \frac{17^n}{n x^n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{|x|}{17}$$
$$= \frac{|x|}{17} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x|}{17} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{|x|}{17}(1+0) = \frac{|x|}{17}$$

It now follows by the Ratio Test that the series converges absolutely when $\frac{|x|}{17} < 1$, *i.e.* when $|x| < 17$, and diverges when $\frac{|x|}{17} > 1$, *i.e.* when $|x| > 17$, so the radius of convergence of this power series is $R = 17$.

It remains to determine what happens when $|x| = 17$, *i.e.* when $x = \pm 17$, where the Ratio Test is inconclusive. At $x = 17$, the series is $\sum_{n=0}^{\infty} \frac{n}{17^n} 17^n = \sum_{n=0}^{\infty} n$; this diverges by the Divergence Test because $\lim_{n\to\infty} n = \infty \neq 0$. At $x = 17$, the series is $\sum_{n=0}^{\infty} \frac{n}{17^n} (-17)^n = \sum_{n=0}^{\infty} (-1)^n n$; this also diverges by the Divergence Test because $\lim_{n\to\infty} (-1)^n n$ does not exist since the even terms head to ∞ and the odd terms head to $-\infty$. It follows that the interval of convergence of the given power series is $(-17, 17)$. □

b. *Ratio Test.* It's not a power series, but let's give the Ratio Test a try anyway:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}((n+1)!)^2}{(2(n+1))!}}{\frac{(-1)^n(n!)^2}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\left((n+1)!\right)^2}{(2(n+1))!}\cdot\frac{(2n)!}{(-1)^n\left(n!\right)^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{\left((n+1)!\right)^2}{\left(n!\right)^2}\cdot\frac{(2n)!}{(2(n+1))!}\right| \\
&= \lim_{n\to\infty}\left|(-1)\cdot\left(\frac{(n+1)\cdot n!}{n!}\right)^2\cdot\frac{(2n)!}{(2n+2)(2n+1)\cdot(2n)!}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty}\frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+\frac{6}{n}+\frac{2}{n^2}} = \frac{1+0+0}{4+0+0} = \frac{1}{4}
\end{aligned}$$

Since the limit is $\frac{1}{4} < 1$, the Ratio Test tells us that the series converges absolutely. □

b. *(Basic) Comparison Test.* We will show that the given series converges absolutely by checking that the corresponding series of positive terms, $\displaystyle\sum_{n=0}^{\infty}\frac{\left(n!\right)^2}{(2n)!}$, converges using the basic form of the Comparison Test.

Observe that for $n \geq 1$ we have

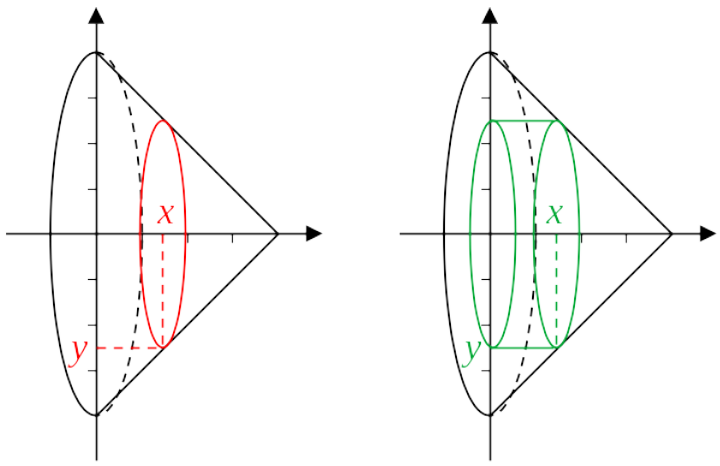
$$\begin{aligned}
0 \leq \frac{\left(n!\right)^2}{(2n)!} &= \frac{n!\cdot n!}{(2n)!} = \frac{n(n-1)\cdots 2\cdot 1\cdot n(n-1)\cdots 2\cdot 1}{(2n)(2n-1)\cdots(n+1)n(n-1)\cdots 2\cdot 1} \\
&= \frac{n(n-1)\cdots 2\cdot 1}{(2n)(2n-2)\cdots 4\cdot 2}\cdot\frac{n(n-1)\cdots 2\cdot 1}{(2n-1)(2n-3)\cdots 3\cdot 1} \\
&= \left(\frac{n}{2n}\cdot\frac{n-1}{2(n-1)}\cdots\frac{2}{4}\cdot\frac{1}{2}\right)\left(\frac{n}{2n-1}\cdot\frac{n-1}{2n-3}\cdots\frac{2}{3}\cdot\frac{1}{1}\right) \\
&\leq \left(\frac{1}{2}\cdot\frac{1}{2}\cdots\frac{1}{2}\cdot\frac{1}{2}\right)(1\cdot 1\cdots 1\cdot 1) = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}
\end{aligned}$$

The Basic Comparison Test now tells us that $\displaystyle\sum_{n=0}^{\infty}\frac{\left(n!\right)^2}{(2n)!} = 1 + \sum_{n=1}^{\infty}\frac{\left(n!\right)^2}{(2n)!}$ converges if the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$ does, which it does because it is a geometric series with common ratio $r = \frac{1}{2}$ and $|r| = \frac{1}{2} < 1$. It follows that the original series converges absolutely. □

c. *Volume Formula.* The region in question is the isosceles right triangle in the fourth quadrant with vertices $(0, 0)$, $(0, -4)$, and $(4, 0)$. When this region is revolved about the

x-axis, part of which is the upper side of the triangle, one obtains a cone with radius $r = 4$ at the blunt end and height $h = 4$. (See the sketches for the Disk/Washer and Cylindrical Shell methods below if you need some help visualizing this.) Plugging these into the volume formula obtained in class gives us a volume of $V = \frac{\pi r^2 h}{3} = \frac{\pi \cdot 4^2 \cdot 4}{3} = \frac{64\pi}{3}$. □

SKETCHES. Here are two sketches of the solid in **c**, one with a disk cross-section drawn in and one with a cylindrical shell cross-section drawn in:



c. *Disks/Washers.* Since the solid was obtained by revolving a region about the x-axis, the cross-sections perpendicular to the axis will be disks and/or washers, and we should use x as the variable. Note that $0 \leq x \leq 4$ for the given region.

Observe that that the cross-section at x is a disk with radius $r = 0 - y = 0 - (4 - x) = x - 4$ and hence area $A(x) = \pi r^2 = \pi(x-4)^2$. It follows that the volume of the solid is given by:

$$V = \int_0^4 \pi r^2 \, dx = \int_0^4 \pi (x-4)^2 \, dx = \pi \int_0^4 \left(x^2 - 8x + 16\right) \, dx = \pi \left[\frac{x^3}{3} - 8\frac{x^2}{2} + 16x\right]_0^4$$
$$= \pi \left[\frac{x^3}{3} - 4x^2 + 16x\right]_0^4 = \pi \left[\frac{4^3}{3} - 4 \cdot 4^2 + 16 \cdot 4\right] - \pi \left[\frac{0^3}{3} - 4 \cdot 0^2 + 16 \cdot 0\right]$$
$$= \pi \left[\frac{64}{3} - 64 + 64\right] - \pi[0 - 0 + 0] = \frac{64}{3}\pi - 0 = \frac{64\pi}{3} \qquad \square$$

c. *Cylindrical Shells.* Since the solid was obtained by revolving a region about the x-axis, the cross-sections parallel to the axis will be cylindrical shells which are perpendicular to the y-axis, so we should use y as the variable. Note that $-4 \leq y \leq 0$ for the given region.

Observe that the cross-section at y is a cylinder with radius $r = 0 - y = -y$ and height $h = x - 0 = y + 4$ (as $y = x - 4$ implies that $x = y + 4$) and hence area $A(y) = 2\pi r h = 2\pi(-y)(y+4)$. (The cylinder is a hollow shell, so its ends do not contribute any area.) It follows that the area of the solid is given by:

$$V = \int_{-4}^{0} 2\pi r h \, dy = \int_{-4}^{0} 2\pi(-y)(y+4) \, dy = 2\pi \int_{-4}^{0} \left(-y^2 - 4y\right) \, dy = 2\pi \left[-\frac{y^3}{3} - 4\frac{y^2}{2}\right]_{-4}^{0}$$

$$= 2\pi \left[-\frac{y^3}{3} - 2y^2\right]_{-4}^{0} = 2\pi \left[-\frac{0^3}{3} - 2 \cdot 0^2\right] - 2\pi \left[-\frac{(-4)^3}{3} - 2 \cdot (-4)^2\right]$$

$$= 2\pi \cdot 0 - 2\pi \left[-\frac{-64}{3} - 32\right] = 0 - 2\pi \left[\frac{64}{3} - \frac{96}{3}\right] = 0 - 2\pi \left[-\frac{32}{3}\right] = \frac{64\pi}{3} \qquad \square$$

d. We will plug $a = 0$, $b = 2$, and $f(x) = x$ into the Left-Hand Rule formula

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + (i-1)\frac{b-a}{n}\right) = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right)$$

and then try to compute the limit. The summation formulas $\sum_{i=1}^{n} 1 = 1+1+1+\cdots+1 = n$ and $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$ will get used along the way.

$$\int_0^2 x\, dx = \lim_{n\to\infty} \frac{2-0}{n} \sum_{i=1}^{n} \left[0 + (i-1)\frac{2-0}{n}\right] = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} (i-1)\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \cdot \frac{2}{n} \sum_{i=1}^{n} (i-1) = \lim_{n\to\infty} \frac{4}{n^2} \left(\left[\sum_{i=1}^{n} i\right] + \left[\sum_{i=1}^{n} 1\right]\right)$$

$$= \lim_{n\to\infty} \frac{4}{n^2} \left(\frac{n(n+1)}{2} + n\right) = \lim_{n\to\infty} \frac{4}{n^2} \left(\frac{n^2+n}{2} + n\right)$$

$$= \lim_{n\to\infty} \frac{4}{n^2} \left(\frac{n^2}{2} + \frac{3n}{2}\right) = \lim_{n\to\infty} \left(\frac{4}{n^2} \cdot \frac{n^2}{2} + \frac{4}{n^2} \cdot \frac{3n}{2}\right)$$

$$= \lim_{n\to\infty} \left(2 + \frac{6}{n}\right) = (2+0) = 2 \qquad \square$$

e. This is a telescoping series since $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. We can use partial fractions to get this decomposition:

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1} = \frac{A(k+1) + Bk}{k(k+1)} = \frac{(A+B)k + A}{k(k+1)}$$

Comparing coefficients in the numerators at the beginning and the end, we see that we must have $A+B = 0$ and $A = 1$, from which it follows that $B = -1$, and so $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.

A little regrouping will now sum the telescoping series:

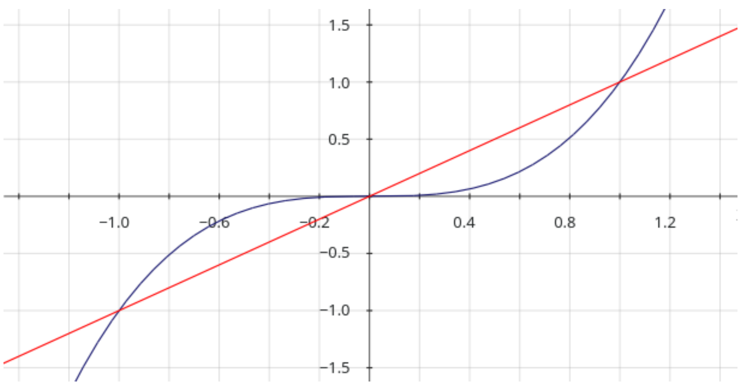
$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+1}\right] = \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \cdots$$
$$= 1 + \left[-\frac{1}{2} + \frac{1}{2}\right] + \left[-\frac{1}{3} + \frac{1}{3}\right] + \left[-\frac{1}{4} + \frac{1}{4}\right] + \cdots = 1 + 0 + 0 + \cdots = 1 \quad \square$$

f. The key here is to use the given information to figure out what the specified region looks like. First, we need to find all the points where the two curves forming its boundary, $y = x$ and $y = x^3$, intersect. Intersection points will occur for those values of x for which the corresponding values of y are equal, *i.e.* when $x = x^3$. Since $x = x^3 = x^2 \cdot x$ can be satisfied when $x = 0$ and $x = \pm 1$ (when $x \neq 0$), the two curves have three points of intersection: $(-1, -1)$, $(0, 0)$, and $(1, 1)$.

Second, it's not hard to see that for $x < -1$ in one direction and $x > 1$ in the other direction, $y = x$ and $y = x^3$ tend away from each other because $y = x^3$ has derivative, namely $3x^2$, greater than the derivative of $y = x$, namely 1. Thus the finite region between the curves is between $x = -1$ and $x = 1$.

Third, since there is a point of intersection at $x = 0$, between $x = -1$ and $x = 1$, we need to check which curve is above the other between $x = -1$ and $x = 0$, and also which is above the other between $x = 0$ amd $x = 1$. We can do this by testing points between the points of intersection. As $\left(\frac{-1}{2}\right)^3 = -\frac{1}{8} > -\frac{1}{2}$, $y = x^3$ is above $y = x$ for $-1 < x < 0$. Similarly, as $\left(\frac{1}{2}\right)^3 = \frac{1}{8} < \frac{1}{2}$, $y = x^3$ is below $y = x$ for $0 < x < 1$.

To help visualise all this, here is plot of the two curves including all three intersection points:

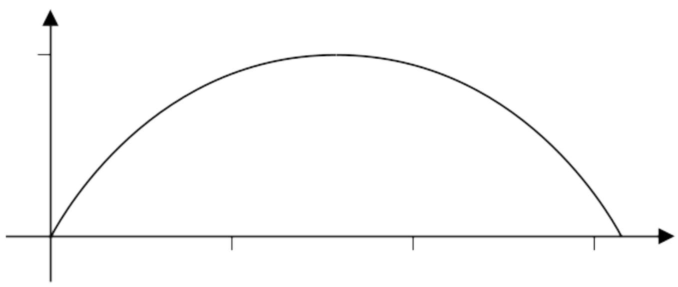


Finally, we put use our knowledge of the region to set up the derivatives that will compute its area and evaluate them:

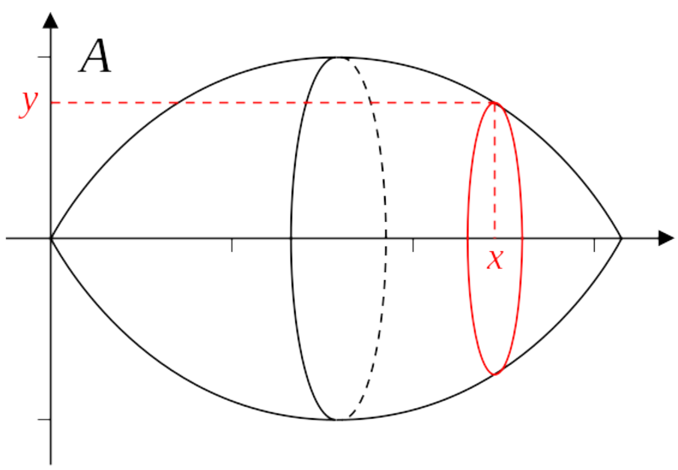
$$
\begin{aligned}
\text{Area} &= \int_{-1}^{1} (\text{upper} - \text{lower})\ dx = \int_{-1}^{0} \left(x^3 - x\right)\ dx + \int_{0}^{1} \left(x - x^3\right)\ dx \\
&= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} \\
&= \left[\frac{0^4}{4} - \frac{0^2}{2}\right] - \left[\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right] + \left[\frac{1^2}{2} - \frac{1^4}{4}\right] - \left[\frac{0^2}{2} - \frac{0^4}{4}\right] \\
&= 0 - \left[\frac{1}{4} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{4}\right] - 0 = -\left[-\frac{1}{4}\right] + \frac{1}{4} = \frac{1}{2} \qquad \blacksquare
\end{aligned}
$$

4. Consider the region between $y = \sin(x)$ and $y = 0$, where $0 \le x \le \pi$. Solid A is obtained by revolving this region about the x-axis and solid B is obtained by revolving the region about the y-axis. Determine which of A and B has greater volume. *[12]*

SOLUTION. Here is a sketch of the original region ...



... and here is one of solid A, the one obtained by revolving the region about the x-axis:



We will find the volume of this solid using the disk/washer method. (If you're tempted to use the cylindrical shell method, you'll have to integrate things like $y \arcsin(y)$, which would be a lot harder.) Since we revolved the region about the x-axis, the disks are perpendicular to the x-axis, so we should use x as the variable. Note that the disk at x has radius $r = y = \sin(x)$ and hence area $A(x) = \pi r^2 = \pi \sin^2(x)$, and that the original region has $0 \le x \le \pi$. We will compute the volume the volume of solid A with the help of the formula $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$.

$$V_A = \int_0^\pi \pi r^2\,dx = \int_0^\pi \sin^2(x)\,dx = \pi \int_0^\pi \frac{1}{2}(1-\cos(2x))\,dx$$

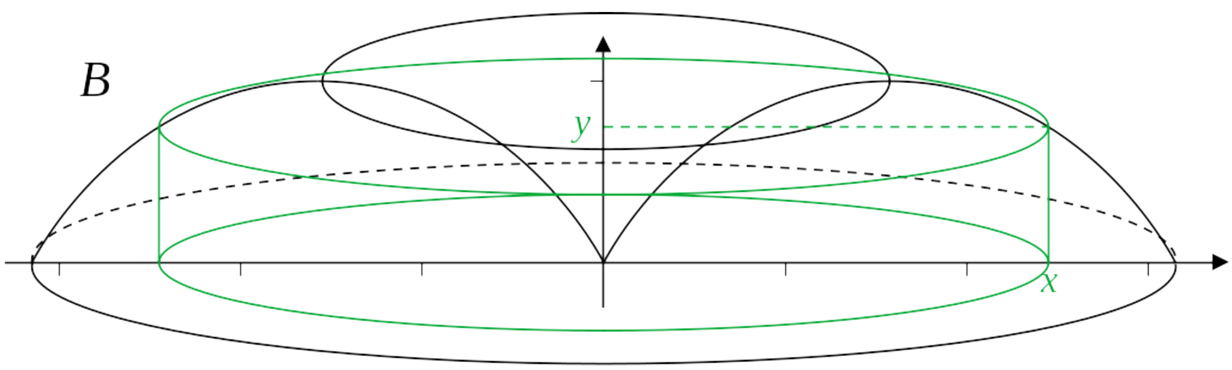
Substitute $u = 2x$ so, $du = 2\,dx$ and $dx = \frac{1}{2}\,du$, and $\begin{matrix} x & 0 & \pi \\ u & 0 & 2\pi \end{matrix}$.

$$= \frac{\pi}{2}\int_0^{2\pi}(1-\cos(u))\,\frac{1}{2}\,du = \frac{\pi}{4}(u - \sin(u))\Big|_0^{2\pi} = \frac{\pi}{4}(2\pi - \sin(2\pi)) - \frac{\pi}{4}(0 - \sin(0))$$

$$= \frac{\pi}{4}(2\pi - 0) - \frac{\pi}{4}(0-0) = \frac{\pi}{4}\cdot 2\pi - 0 = \frac{\pi^2}{2}$$

Thus the volume of solid A is $\dfrac{\pi^2}{2}$.

On to solid B! Here is a sketch of this solid, obtained be revolving the original region about the y-axis:



We will find the volume of this solid using the cylindrical shell method. (If you're tempted to use the disk/washer method, you'll have to integrate things like $x \arcsin(x)$, which would be a lot harder.) Since we revolved the region about the y-axis, the shells are perpendicular to the x-axis, so we should use x as the variable. Note that the cylindrical shell at x has radius $r = x$ and height $h = y = \sin(x)$ and that the original region has $0 \le x \le \pi$. We will compute the volume the volume of solid B with the help of the technique of integration by parts.

$$\begin{aligned} V_B &= \int_0^\pi 2\pi r h\,dx = 2\pi \int_0^\pi x\sin(x)\,dx \qquad \begin{array}{ll} u = x & v' = \sin(x) \\ u' = 1 & v = -\cos(x) \end{array} \\ &= 2\pi \left[-x\cos(x)|_0^\pi - \int_0^\pi 1\,(-\cos(x))\;dx \right] \\ &= 2\pi \left[(-\pi\cos(\pi)) - (-0\cdot\cos(0)) + \int_0^\pi \cos(x)\,dx \right] \\ &= 2\pi \left[(-\pi(-1)) - (-0\cdot 1) + \sin(x)|_0^\pi \right] \\ &= 2\pi\left[\pi - 0 + \sin(\pi) - \sin(0)\right] = 2\pi\left[\pi + 0 - 0\right] = 2\pi^2 \end{aligned}$$

Thus the volume of solid B is $2\pi^2$.

Since $V_A = \dfrac{\pi^2}{2} < 2\pi^2 = V_B$, solid B has greater volume than solid A. ■

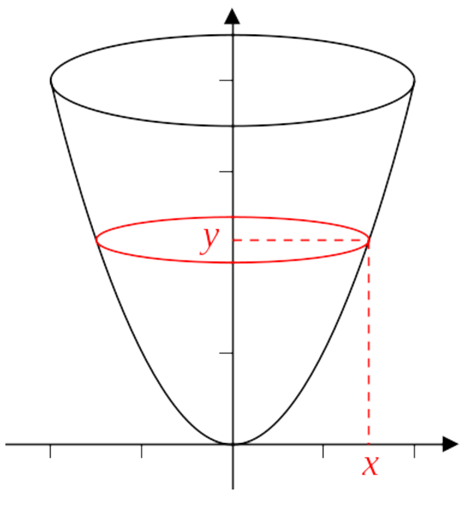
Part Y. Do either *one* (1) of **5** or **6**. *[14]*

5. Consider the curve $y = x^2$, where $0 \le x \le 2$.

a. Find the area of the surface obtained by revolving the curve about the y-axis. *[7]*

b. Find the arc-length of the curve. *[7]*

Solution. **a.** Here is a sketch of the surface:



The given curve is $y = x^2$ for $0 \le x \le 2$. $\dfrac{dy}{dx} = \dfrac{d}{dx}x^2 = 2x$, so the infinitesimal bit of arc-length of the original curve at x is $ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{1 + (2x)^2} = \sqrt{1 + 4x^2}$. This bit of arc-length is revolved about the y-axis through a circle of radius $r = x - 0 = x$.

We plug all this into the area of a surface of revolution integral formula:

$$\text{SA} = \int_0^2 2\pi r\, ds = \int_0^2 2\pi x\sqrt{1+4x^2}\, dx \qquad \begin{array}{l}\text{Substitute } w = 1+4x^2, \text{ so} \\ dw = 8x\,dx \text{ and } x\,dx = \frac{1}{8}\,dw.\end{array}$$

$$= \int_{x=0}^{x=2} 2\pi\sqrt{w}\,\frac{1}{8}\,dw = \frac{\pi}{4}\int_{x=0}^{z=2} w^{1/2}\,dw = \frac{\pi}{4}\cdot\left.\frac{w^{3/2}}{3/2}\right|_{x=0}^{x=2} = \left.\frac{\pi\left(1+4x^2\right)^{3/2}}{6}\right|_0^2$$

$$= \frac{\pi\left(1+4\cdot 2^2\right)^{3/2}}{6} - \frac{\pi\left(1+4\cdot 0^2\right)^{3/2}}{6} = \frac{\pi 17^{3/2}}{6} - \frac{\pi}{6} = \frac{\pi}{6}\left(17\sqrt{17}-1\right) \quad \square$$

b. The given curve is $y = x^2$ for $0 \le x \le 2$. $\frac{dy}{dx} = \frac{d}{dx}x^2 = 2x$, so the infinitesimal bit of arc-length of the original curve at x is $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+(2x)^2} = \sqrt{1+4x^2}$. We plug this into the arc-length of a curve integral formula:

$$\text{AL} = \int_0^2 ds = \int_0^2 \sqrt{1+4x^2}\,dx \qquad \begin{array}{l}\text{Substitute } x = \frac{1}{2}\tan(\theta), \text{ so} \\ dx = \frac{1}{2}\sec^2(\theta)\,d\theta.\end{array}$$

$$= \int_{x=0}^{x=2} \sqrt{1+4\left(\frac{1}{2}\tan(\theta)\right)^2}\,\frac{1}{2}\sec^2(\theta)\,d\theta = \frac{1}{2}\int_{x=0}^{x=2}\sqrt{1+\tan^2(\theta)}\,\sec^2(\theta)\,d\theta$$

$$= \frac{1}{2}\int_{x=0}^{x=2}\sqrt{\sec^2(\theta)}\,\sec^2(\theta)\,d\theta = \frac{1}{2}\int_{x=0}^{x=2}\sec(\theta)\sec^2(\theta)\,d\theta = \frac{1}{2}\int_{x=0}^{x=2}\sec^3(\theta)\,d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln\left(\tan(\theta)+\sec(\theta)\right)\right]_{x=0}^{x=2}$$

$$= \frac{1}{4}\left[2x\sqrt{1+4x^2} + \ln\left(2x+\sqrt{1+4x^2}\right)\right]_0^2$$

$$= \frac{1}{4}\left[2\cdot 2\sqrt{1+4\cdot 2^2} + \ln\left(2\cdot 2+\sqrt{1+4\cdot 2^2}\right)\right]$$

$$\quad - \frac{1}{4}\left[2\cdot 0\sqrt{1+4\cdot 0^2} + \ln\left(2\cdot 0+\sqrt{1+4\cdot 0^2}\right)\right]$$

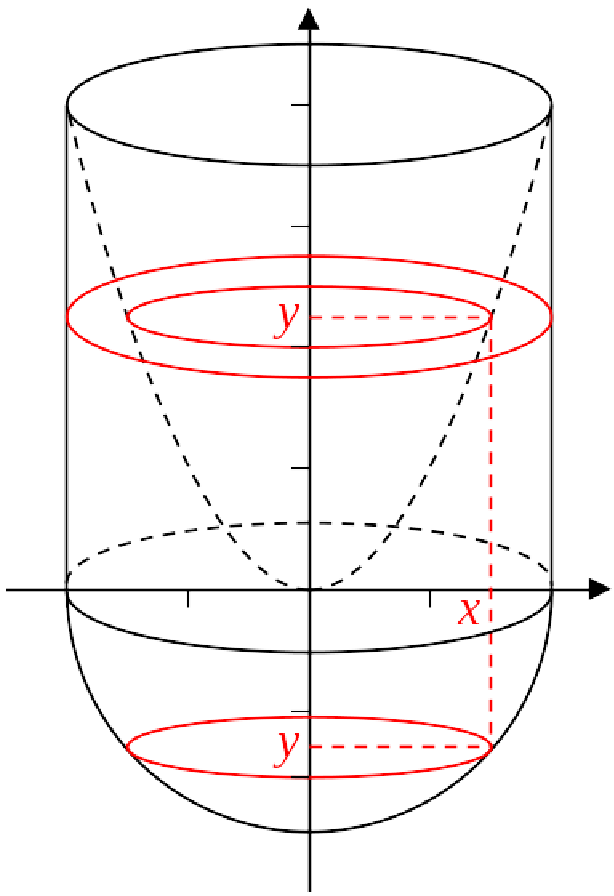
$$= \frac{1}{4}\left[4\sqrt{17} + \ln\left(4+\sqrt{17}\right)\right] - \frac{1}{4}\left[0+\ln(1)\right]$$

$$= \frac{1}{4}\left[4\sqrt{17} + \ln\left(4+\sqrt{17}\right)\right] - \frac{1}{4}\cdot 0 = \sqrt{17} + \frac{1}{4}\ln\left(4+\sqrt{17}\right) \quad \blacksquare$$

6. A solid is obtained by revolving the region below $y = x^2$ and above $y = -\sqrt{4-x^2}$, where $0 \le x \le 2$, about the y-axis. Sketch this solid and find its volume. *[14]*

Solution. $y = x^2$ for $0 \le x \le 2$ is a piece of a hopefully familiar parabola, while $y = -\sqrt{4-x^2}$ for $0 \le x \le 2$ is the part in the fourth quadrant of the circle of radius 2 centred at the origin. Note that $-\sqrt{4-x^2} < x^2$ for all x with $0 \le x \le 2$. We will compute the volume of the solid obtained by revolving the region between the two curves both using the disk/washer method and the cylindrical shell method.

i. Using the disk/washer method. Here is a sketch of the solid, with a couple of disk/washer cross-sections drawn in:



Since the cross-sections are perpendicular to the y-axis, we should use y as the variable. In terms of y, the original region runs from $y = -2$ (when $x = 0$ in $y = -\sqrt{4 - x^2}$) to $y = 4$ (when $x = 2$ in $y = x^2$). However, the integral will need to be broken up because for $-2 \leq y \leq 0$ the cross-sections are disks with radius $R_1 = x = \sqrt{4 - y^2}$ (which follows from $y = -\sqrt{4 - x^2}$ via $x^2 + y^2 = 4$), and hence area

$$A(y) = \pi R_1^2 = \pi \left(\sqrt{4 - y^2} \right)^2 = 4 - y^2 \,,$$

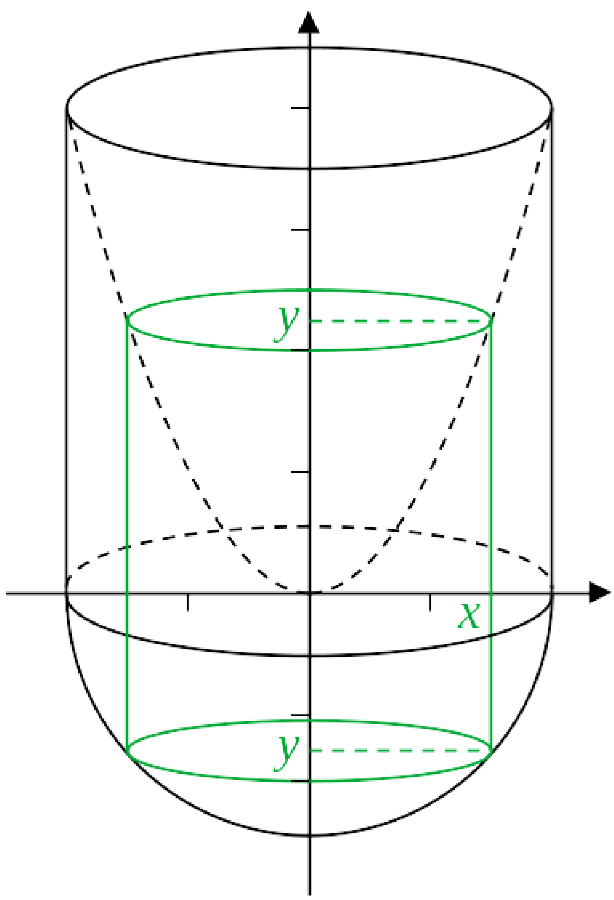
while for $0 \leq y \leq 4$ the cross-sections are washers with outer radius $R_2 = 2$ and inner radius $r_2 = x = \sqrt{y}$ (which follows from $y = x^2$), and hence area

$$A(y) = \pi \left(R_2^2 - r_2^2 \right) = \pi \left(2^2 - (\sqrt{y})^2 \right) = \pi(4 - y) \,.$$

The volume of the solid is thus given by:

$$
\begin{aligned}
V &= \int_{-2}^{4} A(y)\,dy = \int_{-2}^{0} \pi R_1^2\,dy + \int_0^4 \pi\left(R_2^2 - r_2^2\right)\,dy \\
&= \pi\int_{-2}^{0}\left(4-y^2\right)\,dy + \pi\int_0^4 (4-y)\,dy = \pi\left[4y - \frac{y^3}{3}\right]_{-2}^{0} + \pi\left[4y - \frac{y^2}{2}\right]_0^4 \\
&= \pi\left[4\cdot 0 - \frac{0^3}{3}\right] - \pi\left[4\cdot(-2) - \frac{(-2)^3}{3}\right] + \pi\left[4\cdot 4 - \frac{4^2}{2}\right] - \pi\left[4\cdot 0 - \frac{0^2}{2}\right] \\
&= \pi\cdot 0 - \pi\left[-8 - \frac{-8}{3}\right] + \pi\,[16-8] - \pi\cdot 0 = 0 - \pi\left[-\frac{16}{3}\right] + \pi\cdot 8 - 0 \\
&= \frac{16\pi}{3} + \frac{24\pi}{3} = \frac{40\pi}{3} \qquad \square
\end{aligned}
$$

ii. Using the cylindrical shell method. Here is a sketch of the solid, with a cylindrical shell cross-section drawn in:



Since the cross-sections are perpendicular to the x-axis, we should use x as the variable. In terms of x, the original region runs from $x = 0$ to $x = 2$, with the height of the shell at

x being given by $h = x^2 - \left(-\sqrt{4-x^2}\right) = x^2 + \sqrt{4-x^2}$ and its radius by $r = x$. The area of the cylindrical shell at x is thus

$$A(x) = 2\pi rh = 2\pi\left(x^2 + \sqrt{4-x^2}\right)x = 2\pi x^3 + 2\pi x\sqrt{4-x^2}\,.$$

Note that we will have to break up the volume integral according to the dissimilar parts of the integrand. The first part is very easy to handle using the Power Rule and we will handle the second part with the help of the substitution $u = 4 - x^2$, so $du = -2x\,dx$ and $x\,dx = \left(-\frac{1}{2}\right)du$, changing limits as we go: $\begin{matrix} x & 0 & 2 \\ u & 4 & 0 \end{matrix}$. The volume of the solid is thus:

$$\begin{aligned}
V &= \int_0^2 2\pi rh\,dx = \int_0^2 2\pi x\left(x^2 + \sqrt{4-x^2}\right)dx = 2\pi\int_0^2 x^3\,dx + 2\pi\int_0^2 x\sqrt{4-x^2}\,dx \\
&= 2\pi\frac{x^4}{4}\Big|_0^2 + 2\pi\int_4^0 \sqrt{u}\left(1\frac{1}{2}\right)du = 2\pi\frac{2^4}{4} - 2\pi\frac{0^4}{4} - \pi\int_4^0 u^{1/2}\,du \\
&= 2\pi\frac{16}{4} - 2\pi\cdot 0 + \pi\int_0^4 u^{1/2}\,du = 2\pi\cdot 4 - 0 + \pi\frac{u^{3/2}}{3/2}\Big|_0^4 = 8\pi + \frac{2\pi u^{3/2}}{3}\Big|_0^4 \\
&= 8\pi + \frac{2\pi 4^{3/2}}{3} - \frac{2\pi 0^{3/2}}{3} = 8\pi + \frac{2\pi\cdot 8}{3} - 0 = \frac{24\pi}{3} + \frac{16\pi}{3} = \frac{40\pi}{3} \qquad \blacksquare
\end{aligned}$$

Part Z. Do either *one* (1) of **7** or **8**. *[14]*

7. Recall that $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. Find the Taylor series at 0 of $\cosh(x)$

a. using Taylor's formula, *[9]* and

b. without using Taylor's formula. *[5]*

SOLUTION. **a.** Recall that $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$. It follows that

$$\begin{aligned}
\frac{d}{dx}\cosh(x) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh(x) \quad \text{and} \\
\frac{d}{dx}\sinh(x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \sinh(x)\,.
\end{aligned}$$

We will also need to know that $\cosh(0) = \dfrac{e^0 + e^{-0}}{2} = \dfrac{1+1}{2} = 1$ and that $\sinh(0) = \dfrac{e^0 - e^{-0}}{2} = \dfrac{1-1}{2} = 0$. With all of this in hand, we can work out the coefficients of the

Taylor series at 0 of $\cosh(x)$. Working out the successive derivatives of $\cosh(x)$ at 0,

$$\begin{array}{ccc} n & f^{(n)}(x) & f^{(n)}(0) \\ 0 & \cosh(x) & 1 \\ 1 & \sinh(x) & 0 \\ 2 & \cosh(x) & 1 \\ 3 & \sinh(x) & 0 \\ 4 & \cosh(x) & 1 \\ 5 & \sinh(x) & 0 \\ \vdots & \vdots & \vdots, \end{array}$$

it is pretty obvious that the nth derivative of $\cosh(x)$ at 0 is 1 when n is even and 0 when n is odd. Plugging this fact into Taylor's formula tells us that the Taylor series of $f(x) = \cosh(x)$ at 0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} . \quad \square$$

b. We will exploit the fact that $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and do some algebra to write out $\cosh(x)$ as a power series:

$$\begin{aligned} \cosh(x) &= \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(e^x + e^{-x}\right) = \frac{1}{2}\left(\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right] + \left[\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right]\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n + (-x)^n}{n!} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{n!} \cdot \begin{cases} x^n + x^n & \text{if } n \text{ is even} \\ x^n - x^n & \text{if } n \text{ is odd} \end{cases} \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{n!} \cdot \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2x^n & \text{if } n \text{ is even} \end{cases} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{2x^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \end{aligned}$$

As the power series $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ is equal to $\cosh(x)$, it is the Taylor series at 0 of $\cosh(x)$. ■

8. Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n-2}{n!} x^n = -2 - x + \frac{x^3}{6} + \frac{x^4}{12} + \cdots$.

- **a.** Find the radius and interval of convergence of this power series. *[8]*
- **b.** Figure out what function has this power series as its Taylor series. *[6]*

SOLUTION. **a.** As usual, we first try the Ratio Test to discover the radius of convergence of the power series.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(n+1)-2}{(n+1)!}x^{n+1}}{\frac{n-2}{n!}x^n}\right| = \lim_{n\to\infty}\left|\frac{(n-1)x^{n+1}}{(n+1)!}\cdot\frac{n!}{(n-2)x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n-1)x^{n+1}}{(n-2)x^n}\cdot\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{(n-1)x}{n-2}\cdot\frac{1}{n+1}\right| \\
&= \lim_{n\to\infty}\frac{(n-1)|x|}{n^2-n-2} = |x|\cdot\lim_{n\to\infty}\frac{n-1}{n^2-n-2}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} \\
&= |x|\cdot\lim_{n\to\infty}\frac{1-\frac{1}{n}}{n-1-\frac{2}{n}}\begin{matrix}\to 1\\ \to\infty\end{matrix} = |x|\cdot 0 = 0
\end{aligned}$$

Since the limit in the Ratio Test is $0 < 1$ for all x, the series converges for all x, so its radius of convergence is $R = \infty$ and its interval of convergence is $I = (-\infty, \infty)$. □

b. We will exploit the fact that $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and do some algebra:

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{n-2}{n!}x^n &= \sum_{n=0}^{\infty}\left[\frac{n}{n!}-\frac{2}{n!}\right]x^n = \sum_{n=0}^{\infty}\left[\frac{n}{n!}x^n-\frac{2}{n!}x^n\right] = \left[\sum_{n=0}^{\infty}\frac{n}{n!}x^n\right]-\left[\sum_{n=0}^{\infty}\frac{2}{n!}x^n\right] \\
&= \left[\sum_{n=1}^{\infty}\frac{nx^n}{n!}\right]-2\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}\right] = \left[\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}\right]-2e^x = \left[\sum_{n=1}^{\infty}\frac{x\cdot x^{n-1}}{(n-1)!}\right]-2e^x \\
&= x\left[\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}\right]-2e^x = x\left[\sum_{k=0}^{\infty}\frac{x^k}{k!}\right]-2e^x = xe^x-2e^x = (x-2)e^x
\end{aligned}$$

Thus the given power series is equal to the function $f(x) = (x-2)e^x$, and hence the given power series is the Taylor series at 0 of this function. ■

[Total = 100]

Part W. Bonus problems! If you feel like it and have the time, do one or both of these.

9. Consider the following answers to a multiple-choice question:

a. The answer is *b*.
b. The answer is *c*.
c. The answer is *d*.
d. The answer is *e*.
e. None of the above.

Irrespective of the question, what should a student faced with this do? Explain! *[1]*

Solution. Any of **a**–**d** inexorably lead to **e**, but **e** denies the correctness of all of **a**–**d**, and hence of itself too. This is obviously a contradiction … and it means that all of the possible answers are wrong. Rational responses by a student faced with this set of answers would be to ask their instructor or invigilator, or, if that is impossible or produces no useful answer, move on. It's not worth the time or stressed neurons to do more. :-) ■

10. Write a haiku (or several :-) touching on calculus or mathematics in general. *[1]*

What is a haiku?

seventeen in three:
five and seven and five of
syllables in lines

SOLUTION. You're on your own here! :-) ■

ENJOY YOUR SUMMER!

P.S.: You can keep this question sheet. (Paper airplane, fire starter, the possibilities are endless! :-) The solutions to this exam will be posted to the course archive page at `http://euclid.trentu.ca/math/sb/1120H/` *in late April or early May.*