Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2022

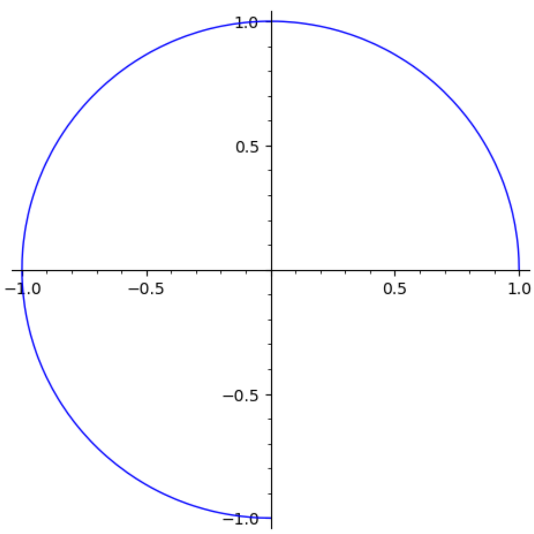
Assignment #6
Arc-length of a parametric curves
Due on Friday, 4 March.[†]
(May be submitted on paper or via Blackboard.[*]*)*

Please show all your work. As with all the assignments in this course, unless stated otherwise on the assignment, you are permitted to work together and look things up, so long as you acknowledge the sources you used and the people you worked with.

One way to describe or define a curve in two dimensions is by way of *parametric equations*, $x = f(t)$ and $y = g(t)$, where the x and y coordinates of points on the curve are simultaneously specified by plugging a third variable, called the *parameter* (in this case t), into functions $f(t)$ and $g(t)$. This approach can come in handy for situations where it is impossible to describe all of a curve as the graph of a function of x (or of y) and arises pretty naturally in various physics problems. (Think of specifying, say, the position (x, y) of a moving particle at time t.)

For a simple example, consider $x = \cos(t)$ and $y = \sin(t)$, for $0 \leq x \leq \dfrac{3\pi}{2}$. This gives the three quarters of the unit circle centred at the origin, namely the parts of the unit circle that are in the first three quadrants. Here is a plot of the curve, as drawn by `SageMath`:

```
sage:  var("y")
y
sage:  var("t")
t
sage:  parametric_plot( (cos(t), sin(t)), (t,0,3*pi/2) )
```



[†] March Fo[u]rth is the only day of the year that doubles as a command! :-)

[*] All else failing, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`

Note that this curve cannot be the graph of a single function of the form $y = f(x)$ because it fails the vertical line test, though it could be broken up into pieces which could each be so described.

1. Suppose a parametric curve is given by $x = f(t)$ and $y = g(t)$, where $a \leq t \leq b$. Explain why the arc-length of this curve is given by $\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$. *[3]*

Hint: Reasoning similar to that given in the lecture on arc-length to justify why the arc-length of the curve $y = f(x)$, $c \leq x \leq d$, is $\int_c^d \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ will do the trick.

2. Use the arc-length formula for parametric curves from **1** to compute the length of the three-quarters of a circle given by $x = \cos(t)$ and $y = \sin(t)$, where $0 \leq x \leq \frac{3\pi}{2}$. Check your answer without using calculus! *[2]*

3. Plot the spiral given by $x = e^t \cos(t)$ and $y = e^t \sin(t)$, where $-2\pi \leq t \leq 2\pi$. You may use `SageMath` or other software to do so, if you want to. *[2]*

4. Use the arc-length formula for parametric curves from **1** to compute the length of the spiral given by $x = e^t \cos(t)$ and $y = e^t \sin(t)$, where $-2\pi \leq t \leq 2\pi$. *[3]*