Taylor Series IV

Tricks and Tips

Basic Idea:

1) if a power series is equal to a function, then that power series is it's taylor series

ex: $\frac{1}{1-x} = 1+x+x^2+x^3+\ldots$

2) if we do unto the Taylor series as we do to the function, the new series is the taylor series of the new function

ex: $\int \frac{1}{1-x}\,dx = \ln(1-x)$

$\int (1+x+x^2+x^3)\,dx$

$= C + x + \frac{x^2}{2} + \frac{x^3}{3}$ & then $C=0$ by plugging in $x=0$ on both sides

ex: Say we want the Taylor Series of $f(x) = \frac{1}{(1-x)^2}$

$$f(x) = \frac{1}{(1-x)^2} = \left[\frac{1}{1-x}\right]^2$$

$$= [1+x+x^2+x^3+x^4+\ldots]^2$$

$$= [1+x+x^2+x^3+x^4+\ldots][1+x+x^2+x^3+x^4+\ldots]$$

$$\begin{aligned} = 1 + x + x^2 + x^3 + x^4 + \ldots \\ + x + x^2 + x^3 + x^4 + \ldots \\ + x^2 + x^3 + x^4 + \ldots \\ + x^3 + x^4 + \ldots \\ + \ldots \end{aligned}$$

$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n$$

Another way:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1-x)^{-1} = \cancel{(-1)}(1-x)^{-2} \cdot \cancel{(-1)} = \frac{1}{(1-x)^2} = f(x)$$

so $f(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1+x+x^2+x^3+\cdots\right)$

$= \left(0+1+2x+3x^2+4x^3+\cdots\right) = \sum_{n=0}^{\infty}(n+1)x^n$

ex: $g(x) = \frac{e^x - 1}{x}$, Find the taylor series of $g(x)$

We know that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

so

$g(x) = \frac{(\cancel{1} + \cancel{x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) \cancel{-1}}{\cancel{x}}$

$= \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots\right)$

$= \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}$

ex: Find the sum of $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$ [ie f(x) equal to this series]

We have stuff related to the exponent in the denominator... maybe we did (or someone did) some integration.

$\int x^n dx = \frac{x^{n+1}}{(n+1)}$ or $\int x^{n-1} dx = \frac{x^n}{n}$

$\int \frac{1}{1-x} dx = -\ln(1-x)$

$= \int (1+x+x^2+x^3+\cdots) dx = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ Plug in $x=0$ and get $C=0$

so $-\ln(1-x)$

$= x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n}$ → To get those n+1's in the denominators, integrate again

$\int \left(\sum_{n=0}^{\infty} \frac{x^n}{n}\right) dx = \sum_{n=0}^{\infty} \int \frac{x^n}{n} dx$

$= K + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n-1)n}$

$$= K + x\sum_{n=0}^{\infty}\frac{x^n}{n(n-1)}$$

$$\int -\ln(1-x)dx \qquad u = 1-x \qquad du = -1dx$$

$$= \int \ln(u)du \quad = \int (1)(\ln(u))\, du \rightarrow \text{Parts:} \quad s = \ln(u) \quad t' = 1$$

$$s' = \frac{1}{u} \qquad t = u$$

$$= u\cdot\ln(u) - \int \frac{1}{\cancel{u}}\cdot\cancel{u}\,du$$

$$= u\ln(u) - u + J$$ • J is a constant

$$= (1-x)\ln(1-x) - (1-x) + J$$

$$= (1-x)\ln(1-x) + x - 1 + J$$

So we have

$$(1-x)\ln(1-x) + x - 1 + J = K + x\sum_{n=0}^{\infty}\frac{x^n}{n(n+1)}$$

$$L = J - K$$

$$= (1-x)\ln(1-x) + x - 1 + L = x\sum_{n=0}^{\infty}\frac{x^n}{n(n+1)}$$ → plug in $x=0$, solve for constant

$$(1-0)\ln(1-0) + 0 - 1 + L = 0\sum_{n=0}^{\infty}\frac{0^n}{n(n+1)}$$

$$-1 + L = 0$$

$$L = 1$$

Thus

$$= (1-x)\ln(1-x) + x \cancel{-1} \cancel{+1} = x\sum_{n=0}^{\infty}\frac{x^n}{n(n+1)}$$

$$= (1-x)\ln(1-x) + x = x\sum_{n=0}^{\infty}\frac{x^n}{n(n+1)}$$

so $$\sum_{n=0}^{\infty}\frac{x^n}{n(n+1)} = \frac{(1-x)\ln(1-x) + x}{x} = \left(\frac{1}{x} - 1\right)\ln(1-x) + 1$$

[Not on exam!

$$\sum_{n=0}^{\infty}\frac{x^n}{n^2} = f(x)$$ & find $f(x)$]

exam level example: $\sum_{n=0}^{\infty} \frac{2^{n+2}}{n!} = ?$

Find $g(x)$
where $g(x)$ is the sum of a series
given the one above for a suitable sum of x

$$\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} \overset{?}{=} g(x) = g(2)$$

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$$

$$= x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= e^x x^2$$

So $\sum_{n=0}^{\infty} \frac{2^{n+2}}{n!} = g(2) = 2^2 e^2 = (2e)^2 = 4e^2$