Lecture 23

Apr. 5th, 2022

If a function equals a power series, that series is the Taylor series.

Basic Idea:

1) If a power series is equal to a function, then that series is the function's Taylor Series

ex/ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

2) If we do unto the Taylor series as we do to the function, the new series is the Taylor series of the new function.

ex/ $\int \frac{1}{1-x}\,dx = -\ln(1-x)$

$\int (1 + x + x^2 + x^3 + \ldots)\,dx = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$

and $C=0$ by plugging in 0 for both sides

$\Rightarrow -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$

ex/ Suppose we want the Taylor series for $f(x) = \frac{1}{(1-x)^2}$.

$$\begin{aligned}
f(x) = \frac{1}{(1-x)^2} &= \left(\frac{1}{1-x}\right)^2 = (1 + x + x^2 + x^3 + \ldots)^2 \\
&= (1 + x + x^2 + x^3 + \ldots)(1 + x + x^2 + x^3 + \ldots) \\
&= (1 + x + x^2 + x^3 + \ldots) + (x + x^2 + x^3 + x^4 + \ldots) + (x^2 + x^3 + x^4 + x^5 + \ldots) \\
&\quad + (x^3 + x^4 + x^5 + x^6 + \ldots) + \ldots \\
&= 1 + 2x + 3x^2 + 4x^3 + \ldots \\
&= \sum_{n=0}^{\infty} (n+1)x^n
\end{aligned}$$

or $\cancel{f(x)}$

$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2} = f(x)$

so $f(x) = \frac{d}{dx}(1 + x + x^2 + x^3 + \ldots) = 0 + 1 + 2x + 3x^2 + 4x^3 + \ldots$

$= \sum_{n=0}^{\infty} (n+1)x^n$

ex/ $g(x) = \frac{e^x - 1}{x}$, Find the Taylor Series.

We know the series for e^x. ($e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$)

$$\text{so } g(x) = \frac{(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots)-1}{x}$$

$$= \frac{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots}{x}$$

$$= 1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

ex/ Find the sum of $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$ (ie. the function equal to this series)

We have stuff related to the exponent in the denominator ... maybe there was integration involved.

$$\int \frac{1}{1-x}\,dx = -\ln(1-x)$$

$$= \int (1+x+x^2+x^3+\dots)\,dx = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

plug in $x=0 \rightarrow C=0$

$$\Rightarrow -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

To get the $(n+1)$s in the denominator: integrate again.

$$\int \sum_{n=1}^{\infty} \frac{x^n}{n}\,dx = \sum_{n=1}^{\infty} \int \frac{x^n}{n}\,dx = K + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = K + x\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$$

and $\int -\ln(1-x)\,dx = \int \ln(u)\,du = \int 1 \cdot \ln(u)\,du$

sub $\boxed{u = 1-x \Rightarrow du = -dx}$ parts $\boxed{s = \ln(u),\ t' = 1,\ s' = \frac{1}{u},\ t = u}$

$$= u\ln(u) - \int \frac{1}{u} \cdot u\,du = u\ln(u) - u + J$$

$$= (1-x)\ln(1-x) - (1-x) + J = (1-x)\ln(1-x) + x - 1 + J$$

$\downarrow$

So we have

$$(1-x)\ln(1-x) + x - 1 + J = K + x\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$$

$$\Rightarrow (1-x)\ln(1-x) + x - 1 + L = x\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$$

Let $x=0$.

$$\cancel{(1-0)\ln(1-0)}^{\to 0} + \cancel{0}^{0} - 1 + L = \cancel{0\cdot\sum_{n=1}^{\infty}\frac{0^n}{n(n+1)}}^{\to 0} \Rightarrow -1+L=0 \Rightarrow L=1$$

$$\Rightarrow (1-x)\ln(1-x) + x = x\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$$

So $$\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)} = \frac{(1-x)\ln(1-x)+x}{x} = \left(\frac{1}{x}-1\right)\ln(1-x) + 1.$$

ex/ Not on Exam!

$$\sum_{n=1}^{\infty}\frac{x^n}{n^2} = f(x), \text{ find } f(x).$$

ex/ $\sum_{n=0}^{\infty}\frac{2^{n+2}}{n!} = ~~f(x)~~ ?$ Find $g(x)$ where $g(x) = \sum_{n=0}^{\infty}\frac{x^{n+2}}{n!}$. (then $g(2)$)

$$g(x) = \sum_{n=0}^{\infty}\frac{x^{n+2}}{n!} = x^2\sum_{n=0}^{\infty}\frac{x^n}{n!} = x^2e^x$$

and $g(2) = (2)^2e^2 = 4e^2$

$$\therefore \sum_{n=1}^{\infty}\frac{2^{n+2}}{n!} = 4e^2.$$