Series VII

Today Summing conditionally convergent series and then the Ratio and Root tests

Let's make the alternating harmonic series,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots, \text{ Sum to } \underline{\ 2\ }. \longrightarrow \text{can pick any number}$$

We'll do this by rearranging the series. (If you don't it will add up to ln(2))

$$2 = \underbrace{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}}_{>2}$$

$$-\frac{1}{2}$$

$$+\frac{1}{17} + \frac{1}{19} + \dots + \frac{1}{2k+1} > 2$$

$$-\frac{1}{4}$$

$$+\frac{1}{2k+1} + \cdots$$

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \cdots = \infty$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \infty$$

Since both diverges

Use the positive terms (in order) that you haven't used yet to get over 2 and then the negative terms to get below ("lather, rinse, repeat")

The partial sums get closer & closer to 2
-every time you use an even term $-\frac{1}{2n}$ you'll be with $\frac{1}{2n}$ of 2

So the limit of partial sums will be 2...

Note that rearranging only finitely many terms dose not change the sum. Infinitely many must be rearranged.

We could do this with any target sum (& any conditionally convergen series)
Absolutly convergent series converge to the same sum no matter how you rearrange them

Ratio Test

Suppose $\sum_{n=0}^{\infty} a_n$ is a series such that (past some point) $a_n \neq 0$

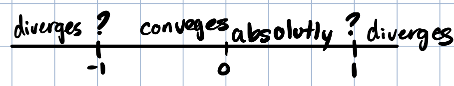
Then if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} <1 & \text{the series converges absolutely} \\ =1 & \text{No Information} \\ >1 & \text{the series diverges} \end{cases}$

ex: For which values of x does $\sum_{n=0}^{\infty} \frac{x^n}{2n+3}$ converge?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{2(n+1)+3}}{\frac{x^n}{2n+3}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{2n+5} \cdot \frac{2n+3}{x^n} \right| = \lim_{n\to\infty} \left| \overset{\text{constant}}{x} \cdot \frac{2n+3}{2n+5} \right|$$

$$= |x| \cdot \lim_{n\to\infty} \frac{2n+3}{2n+5} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = |x| \cdot \lim_{n\to\infty} \frac{2+\frac{3}{n}\to 0}{2+\frac{5}{n}\to 0} = |x| \frac{\cancel{2}+0}{\cancel{2}+0} = |x|$$

so if $|x|<1$, the series converges absolutly
if $|x|>1$, the series diverges
if $|x|=1$, we have to resort to other tests



If $|x|=1$, then $x=-1$ or $x=1$

$x=-1$: then the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+3} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \ldots$

we'll use the alternating series test

1) The series alternates sign because $(-1)^n$ does & $\frac{1}{2n+3} > 0$

2) $\left| \frac{(-1)^{n+1}}{2(n+1)+3} \right| = \frac{1}{2n+5} < \frac{1}{2n+3} = \left| \frac{(-1)^n}{2n+3} \right|$ so the absolute values of the terms are decreasing

3) $\lim_{n\to\infty} \left| \frac{(-1)^n}{2n+3} \right| = \lim_{n\to\infty} \frac{1 \to 1}{2n+3 \to \infty} = 0$

So, by the Alt. Series test $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+3}$ converges

$x=1$: then the series is $\sum_{n=0}^{\infty} \frac{1}{2n+3}$, which diverges by the generized p-Test since $p=1\cdot 0=1\leq 1$

Thus $\sum_{n=0}^{\infty} \frac{x^n}{2n+3}$ converges if $x \in [-1,1)$ and diverges otherwise