Series II

Our first two tests for convergence (or not) of series

- <u>Divergence test</u>

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges (dose not converge)

Why is this true?

$a_0 + a_1 + a_2 + a_3 + \ldots$

if $a_n \to L \neq 0$

$\ldots$ + close to L + close to L + $\ldots$
$\to +\infty$ if $L>0$
$-\infty$ if $L<0$

if a_n heads of to ∞ or $-\infty$, the remaining sums have to do the same...

if the a_n s bounces around to make $\lim_{n\to\infty} a_n$ fail, the partial sums will bounce around too

Alternatively, you can show this by checking that if $\sum_{n=0}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$

Suppose this happens, then $\lim_{n\to\infty} S_n = L$, for some L

ie for any $\varepsilon>0$ there is an N, st if $n \geq N$ then $|S_n - L| < \frac{\varepsilon}{2}$

But then
$$\begin{aligned}|a_n| &= |S_n - S_{n-1}| \\ &= |S_n - L + L - S_{n-1}| \\ &\leq |S_n - L| + |L - S_{n-1}| \\ &= |S_n - L| + |S_{n-1} - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Thus, for any $\varepsilon>0$, there is an N st if $n \geq N$, then $|a_n - 0| = |a_n| < \varepsilon$

ie $\lim_{n\to\infty} a_n = 0$

Sadly, the Divergence Test is of limited use, there are lots of series for which $\lim_{n\to\infty} a_n = 0$ but $\sum_{n=0}^{\infty} a_n$ diverges

eg: $\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

Obviously survives the Divergence Test because $\lim_{n\to\infty} \frac{1}{n} = 0$

but the series adds up to ∞

$$1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}+\underbrace{\frac{1}{9}+\frac{1}{10}+\cdots}$$

$$\geq 1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\frac{8}{16}$$

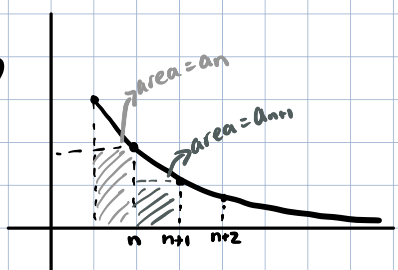
$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots = \infty$$

- **Integral Test**

Suppose $\sum_{n=c}^{\infty} a_n$ comes from a function $f(x)$ via $a_n = f(n) \geq 0$

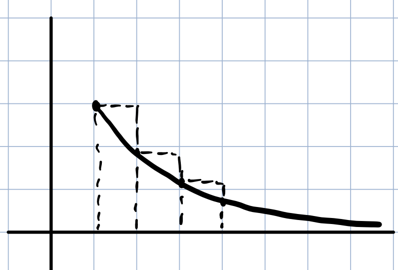
If $f(x)$ is a decreasing function & integrable on $[c,\infty)$,
then $\sum_{n=c}^{\infty} a_n$ converges on diverges
exactly as the improper integral $\int_c^{\infty} f(x)\,dx$ does.

Why?



area of all the rectangles
$= \sum_{n=c}^{\infty} a_n \leq \int_c^{\infty} f(x)\,dx$

If the integral converges, sum dose too



This gives $\sum_{n=c}^{\infty} a_n \geq \int_c^{\infty} f(x)\,dx$, so if $\int_c^{\infty} f(x)$ diverges, so does the sum

eg $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{c\to\infty} \int_1^{c} \frac{1}{x}\,dx = \lim_{c\to\infty} \ln(x)\Big|_1^c$

$$= \lim_{c\to\infty} (\ln(c) - \cancel{\ln(1)}^{\,0})$$

$= \lim_{c\to\infty} \ln(c) = \infty$

eg: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because $\int_1^{\infty} \frac{1}{x^2}\,dx$

$$= \lim_{c\to\infty} \int_1^{c} \frac{1}{x^2}\,dx$$

$$= \lim_{c\to\infty} \int_1^{c} x^{-2}\,dx$$

$$= \lim_{c\to\infty} \left.\frac{x^{-1}}{-1}\right|_1^c$$

$$= \lim_{c\to\infty} \left.\frac{-1}{x}\right|_1^c$$

$$= \lim_{c\to\infty} \left(\frac{-1}{c} - \frac{-1}{1}\right)$$

$$= \lim_{c\to\infty} \left(1 - \frac{1}{c}\right) = 1 - 0 = 1$$

so we know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (but... Didn't get the last sentence