Series I

- A sequence $\{a_n\}$ is a list of numbers indexed by the integers starting from some point.

  ex: $a_n = \frac{1}{n}$ $(n \geq 1)$

  - A sequence $\{a_n\}$ <u>converges</u> (ie has a limit) as $n \to \infty$ means that there is an L and for any $\varepsilon > 0$ we can find an N s.t. for all $n \geq N$, $|a_n - L| < \varepsilon$
  - One useful trick is that if $a_n = f(n)$ for all n & $f(x)$ is a function s.t. $\lim_{x\to\infty} f(x)$ exists, then that limit $= \lim_{n\to\infty} a_n$

- A <u>series</u> $\sum_{n=0}^{\infty} a_n$ is the sum of sequence $\{a_n\}$ <u>if</u> it exists.

  eg: last time we briefly looked at $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$

  eg: $\sum_{n=0}^{\infty} 1 = 1+1+1+1+\ldots$ dose not add up (except maybe to ∞ which is not a real number)

  We need a decent definition of "add up"...

<u>Converges</u> (ie adds up) to a sum $A \in \mathbb{R}$ if it's sequence of partial sums converges to A

The partial sum up to n of the series is $a_0 + a_1 + a_2 + a_3 + a_4 + \ldots + a_n = S_n$

so $\sum_{n=0}^{\infty} a_n$ converges to A if $\lim_{n\to\infty} S_n = A$

eg: In some series the partial sums have nice formulas

eg: $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = \dfrac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}}$ ... but most don't

eg: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2} = ??$

We will therefor develop ways to test series to see if they converge or not with out having to know the sum

Prototypes for summing series: Two cases where the partial sums have nice formulas

1) Geometric series

$\sum_{n=0}^{\infty} ar^n$, where a is the first term and $r = \frac{ar^{n-1}}{ar^n}$ is the <u>common ratio</u>: $\sum_{n=0}^{\infty} ar^i$

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

$$= a(1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n) \qquad \text{\& now what?}$$

Observe that if we multiply this by $1-r$, we get

$$a(1 + r + r^2 + \cdots + r^{n-1} + r^n)(1-r)$$

$$= a\left(\begin{matrix} 1 + \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^{n-1}} + \cancel{r^n} \\ - \cancel{r} - \cancel{r^2} - \cdots - \cancel{r^{n-1}} - \cancel{r^n} \end{matrix}\right)$$

$$= a(1 - r^{n+1})$$

$$\therefore a(1 + r + \cdots + r^n) = a\left(\frac{1 - r^{n+1}}{1 - r}\right)$$

so, $\sum_{n=0}^{\infty} ar^n$ has a limit

<u>if</u> $\lim a\left(\frac{1 - r^{n+1}}{1 - r}\right)$ has a limit

So when does this limit exist?

It converges <u>exactly when</u> either $a = 0$ or $|r| < 1$ ie $-1 < r < 1$

If $|r| < 1$ then $r^{n+1} \to 0$ so the limit is $\frac{a}{1-r}$

On the other hand if $r > 1$, then $r^{n+1} \to \infty$ if $r = 1$, we'd be dividing by 0,

if $r = -1$, then $r^{n+1} = \begin{cases} 1 & \text{if } n \text{ is added} \\ -1 & \text{if } n \text{ is subtracted} \end{cases}$

so there is no limit

if $r < -1$, then r^{n+1} oscillates with ever longer savings, so there is no limit

2) The other easy example: Telescoping series

ie $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{4} + \cdots$

$$S_n = \sum_{i=0}^{n} \frac{1}{(i+1)(i+2)}$$

$$= 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \cdots + \frac{1}{(n+1)(n+2)} \qquad = \text{nice formula!}$$

$$= \frac{A}{i+1} + \frac{B}{i+2}$$

$$= \frac{A(i+2)+B(i+1)}{(i+1)(i+2)}$$

$$= \frac{(A+B)i+(2A+B)}{(i+1)(i+2)}$$

Then $1 = (A+B)i + (2A+B)$

so $A+B=0$

& $2A+B=1$

$\Rightarrow B=-A \Rightarrow 2A-A=1$

$\Rightarrow A=1$ & $B=-1$

$$\therefore \frac{1}{(i+1)(i+2)} = \frac{1}{i+1} - \frac{1}{i+2}$$

thus $S_n = \underbrace{1 \cdot \frac{1}{2}}_{i=0} + \underbrace{\frac{1}{2} \cdot \frac{1}{3}}_{i=1} + \underbrace{\frac{1}{3} \cdot \frac{1}{4}}_{i=2} + \cdots + \underbrace{\frac{1}{(n+1)(n+2)}}_{i=n}$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right)$$

$$= 1 - \frac{1}{n+2}$$

Thus $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$

$$= \lim_{n \to \infty} S_n$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n+2}\right)$$

$$= 1 - 0 \quad = 1$$