Sequences & Series

(Ch. 11 in the textbook)

A sequences is a list of real numbers indexed by the non-negative integers

Index: n	0	1	2	3	4 ...
a_n	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$ ...

In this case, $a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

Dose the sequence have a limit? Yes, it's zero.

Definition: A sequence, $\{a_n\}$, has limits, L (ex: "$\text{Limit}_{n\to\infty} a_n = L$")

means~ for every $\varepsilon > 0$ you can find an $N > 0$, for all $n \geq N$, $|a_n - L| < \varepsilon$

$\text{limit}_{n\to\infty} \frac{1}{2^n} = 0$ Given on $\varepsilon > 0$, we'll reverse-engineer the necessary N from $\left|\frac{1}{2^n} - 0\right| < \varepsilon$

$\left|\frac{1}{2^n} - 0\right| < \varepsilon$

$\Rightarrow \left|\frac{1}{2^n}\right| < \varepsilon$

$\Rightarrow \frac{1}{2^n} < \varepsilon$

$\Rightarrow 1 < \varepsilon(2^n)$

$\Rightarrow \frac{1}{\varepsilon} < 2^n$

$\Rightarrow \log_2\left(\frac{1}{\varepsilon}\right) < n$

If you now let N be any integer $\geq \log_2\left(\frac{1}{\varepsilon}\right)$,

then $n > N \geq \log_2\left(\frac{1}{3}\right)$

so $\frac{1}{\varepsilon} < 2^n$

so $1 < \varepsilon(2^n)$

so $\frac{1}{2^n} < \varepsilon$

so $\left|\frac{1}{2^n} - 0\right| < \varepsilon$ as required to have $\lim_{n\to\infty} \frac{1}{2^n} = 0$

Let's try finding the limit of

$a_n = \dfrac{3^n+4}{3^{n-1}+3}$ So how do we find $\displaystyle\lim_{n\to\infty}\frac{3^n+4}{3^{n-1}+3}$?

$$\Rightarrow \lim_{n\to\infty}\frac{3^n+4}{3^{n-1}+3}\cdot\frac{\frac{1}{3^n}}{\frac{1}{3^n}}$$

$$\Rightarrow \lim_{n\to\infty}\frac{\frac{3^n}{3^n}+\frac{4}{3^n}}{\frac{3^{n-1}}{3^n}+\frac{3}{3^n}}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1+\frac{4}{3^n}}{\frac{1}{3}+\frac{1}{3^{n-1}}}$$

→ zero

→ zero

$$\Rightarrow \lim_{n\to\infty}\frac{1+0}{\frac{1}{3}+0} = \frac{1}{\frac{1}{3}} = 3$$

Limits with a discreate variable (like "n") instead of a continnous variable like x, obey the same rules except where continunity is nessasary.

We can often work around this using the following trick:

Suppose $a_n = f(n)$ where $f(x)$ is continuous (differentiable, etc)

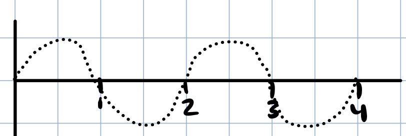
Then we can explait the fact in this case $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x)$ provided that last limit exists

bad example: $a_n = \sin(n\pi)$
$= 0$

for all "n" so $\lim_{n\to\infty} a_n = 0$

on the other hand $\lim_{x\to\infty}\sin(\pi x) = ?$

This doesn't exist



$$a_n = \frac{3^n + 4}{3^{n-1} + 3}$$

$$f(x) = \frac{3^x + 4}{3^{x-1} + 3}$$

$$\lim_{x \to \infty} \frac{3^x + 4}{3^{x-1} + 3}$$

$$\lim_{x \to \infty} \frac{3^x + 4 \to \infty}{3^{x-1} + 3 \to \infty}$$

So we can use l'Hôpital's Rule

$$\lim_{x \to \infty} \frac{\frac{d}{dx} 3^x + 4}{\frac{d}{dx} 3^{x-1} + 3}$$

$$\lim_{x \to \infty} \frac{\cancel{\ln(3)} \cdot 3^x \cancel{+0}}{\cancel{\ln(3)} \cdot 3^{x-1} \cancel{+0}}$$

$$\lim_{x \to \infty} 3 = 3$$

Back to origenal example

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 2$$