Lecture 14

Mar 4^{th}, 2022

Sequence: a list of numbers (real) indexed by the non-negative integers

Index → n :	0	1	2	3	4	...
Sequence → a_n :	1	$1/2$	$1/4$	$1/8$	$1/16$	...

In this case, $a_n = (\frac{1}{2})^n$

Does the sequence have a limit?
Informally, it is easy to see that the limit is 0.
Formally:

* A sequence $\{a_n\}$ has limit L (ie. $\lim_{n\to\infty} a_n = L$) means that for every $\epsilon > 0$, you can find an $N > 0$ such that for all $n \geq N$, $|a_n - L| < \epsilon$

$\lim_{n\to\infty} \frac{1}{2^n} = 0$. Let's check:

Given an $\epsilon > 0$, we'll reverse engineer the necessary N from $|\frac{1}{2^n} - 0| < \epsilon$.

$$\left|\frac{1}{2^n} - 0\right| < \epsilon \Leftrightarrow \left|\frac{1}{2^n}\right| < \epsilon \Leftrightarrow \frac{1}{2^n} < \epsilon$$

$$\Leftrightarrow 1 < 2^n \epsilon \quad \Leftrightarrow \quad \frac{1}{\epsilon} < 2^n \Leftrightarrow \log_2\left(\frac{1}{\epsilon}\right) < n$$

If you now let N be any integer $\geq \log_2(\frac{1}{\epsilon})$,
then $n \geq N \geq \log_2(\frac{1}{\epsilon})$
$\Rightarrow \frac{1}{\epsilon} < 2^n$
$\Rightarrow 1 < \epsilon \cdot 2^n$
$\Rightarrow \frac{1}{2^n} < \epsilon$
$\Rightarrow |\frac{1}{2^n} - 0| < \epsilon$ as is required to have $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

Find the limit of $a_n = \frac{3^n + 4}{3^{n-1} + 3}$

$$\lim_{n\to\infty} \frac{3^n+4}{3^{n-1}+3} = \lim_{n\to\infty} \frac{3^n+4}{3^{n-1}+3} \cdot \frac{\frac{1}{3^n}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{\frac{3^n}{3^n} + \frac{4}{3^n}}{\frac{3^{n-1}}{3^n} + \frac{3}{3^n}}$$

$$= \lim_{n\to\infty} \frac{1 + \cancel{\frac{4}{3^n}}^{\to 0}}{\frac{1}{3} + \cancel{\frac{1}{3^{n-1}}}^{\to 0}} = \lim_{n\to\infty} \frac{1}{1/3} = 3$$

Limits with a discrete variable (ie. n), instead of a continuous variable like x, obey the same rules

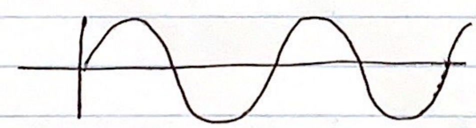
We can often work around this using the following:

Suppose $a_n = f(n)$ where $f(x)$ is continuous/differentiable. We can exploit the fact that in this case:

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x)$ provided that the last limit exists.

ex/ $a_n = \sin(n\pi) = 0$ for all n.
$\Rightarrow \lim_{n\to\infty} a_n = 0$

On the other hand, $\lim_{x\to\infty} \sin(\pi x)$ does not exist.



$\rightarrow$ If $\lim_{n\to\infty} f(n)$ exists, $\lim_{x\to\infty} f(x)$ may or may not exist.
$\rightarrow$ If $\lim_{x\to\infty} f(x)$ exists, $\lim_{n\to\infty} f(n)$ exists and $\lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x)$.

L'Hôpital's Rule

ex/ $a_n = \dfrac{3^n + 4}{3^{n-1} + 3}$, $f(x) = \dfrac{3^x + 4}{3^{x-1} + 3}$

$$\Rightarrow \lim_{n\to\infty} \frac{3^n+4}{3^{n-1}+3} = \lim_{x\to\infty} \frac{3^x+4}{3^{x-1}+3} \quad \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{\frac{d}{dx}(3^x+4)}{\frac{d}{dx}(3^{x-1}+3)}$$

$$= \lim_{x\to\infty} \frac{3^x \cancel{\ln(3)}}{3^{x-1}\cancel{\ln(3)}} = \lim_{x\to\infty} 3 = 3$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots = 2$$

At each step, your distance from 2 is the same as the length of the step you just made.