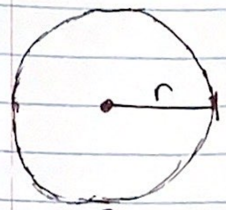
Lecture 13

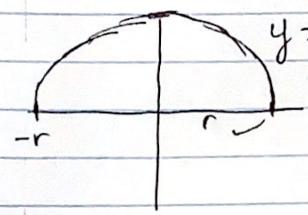
Mar. 1st, 2022




$C = 2\pi r$ $\qquad$ $SA = 4\pi r^2$

$A = \pi r^2$ $\qquad$ $V = \frac{4}{3}\pi r^3$

How do we get $C = 2\pi r$?



$y = \sqrt{r^2 - x^2}$

$a = -r$

$b = r$

$$\frac{dy}{dx} = \frac{1}{2}(r^2 - x^2)^{-1/2} \cdot (-2x)$$
$$= \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\text{Arc-Length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_{-r}^{r} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\, dx$$
$$= \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2-x^2}}\, dx$$
$$= \int_{-r}^{r} \sqrt{\frac{r^2-x^2}{r^2-x^2} + \frac{x^2}{r^2-x^2}}\, dx$$
$$= \int_{-r}^{r} \sqrt{\frac{r^2}{r^2-x^2}}\, dx$$
$$= \int_{-r}^{r} r\sqrt{\frac{1}{r^2-x^2}}\, dx$$
$$= r\int_{-r}^{r} \frac{1}{\sqrt{r^2-x^2}}\, dx$$

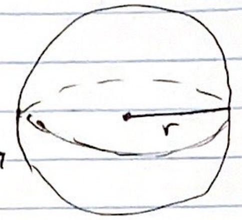
$x = r\sin(\theta) \Rightarrow dx = r\cos(\theta)d\theta$

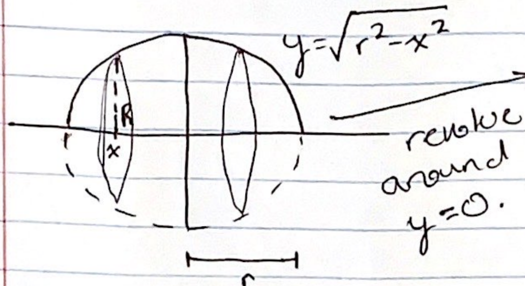
$\hookrightarrow \theta = \arcsin\left(\frac{x}{r}\right)$

$$= r\int_{x=-r}^{x=r} \frac{1}{\sqrt{r^2 - (r\sin(\theta))^2}} \cdot r\cos(\theta)d\theta$$
$$= r\int_{x=-r}^{x=r} \frac{\cancel{r}\cos(\theta)}{\cancel{r}\,\cancel{\sqrt{1-\sin^2(\theta)}}^{\,\cos(\theta)}}\, d\theta$$
$$= r\int_{x=-r}^{x=r} d\theta = r\theta\Big|_{x=-r}^{x=r} = r\arcsin\left(\frac{x}{r}\right)\Big|_{-r}^{r} = r\arcsin(1) - r\arcsin(-1)$$
$$= r\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi r$$ [arc length of semi circle]

Therefore, the circumference of a circle $= 2\cdot(\text{arclength of semicircle}) = 2\pi r.$

$V_{sphere} = \frac{4}{3}\pi r^3$?





Disk Method: $A_{disk} = \pi R^2$, where $R = \sqrt{r^2-x^2} - 0 = \sqrt{r^2-x^2}$.

$$= \pi\sqrt{r^2-x^2}^{\,2} = \pi(r^2-x^2)$$

$$\Rightarrow V = \int_{-r}^{r} A\,dx = \int_{-r}^{r} \pi R^2\,dx = \int_{-r}^{r} \pi(r^2-x^2)\,dx$$

$$= \pi\int_{-r}^{r}(r^2-x^2)\,dx = \pi\left(r^2x - \frac{x^3}{3}\right)\Big|_{-r}^{r}$$

$$= \pi\left(r^2\cdot r - \frac{r^3}{3} - \left(r^2(-r) - \frac{(-r)^3}{3}\right)\right)$$

$$= \pi\left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right)$$

$$= \pi\left(\frac{2r^3}{3} + \frac{2r^3}{3}\right) = \frac{4}{3}\pi r^3$$

Surface area of a circle should be the sum of the circumferences of the disk.

$$SA = \int_{-r}^{r} 2\pi R\,dx = \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\,dx$$

$x = r\sin(\theta)$

$dx = r\cos(\theta)\,d\theta$

x	θ
$-r$	$-\pi/2$
r	$\pi/2$

$$= \int_{-\pi/2}^{\pi/2} 2\pi\sqrt{r^2-(r\sin(\theta))^2}\cdot r\cos(\theta)\,d\theta$$

$$= 2\pi r^2\int_{-\pi/2}^{\pi/2}\sqrt{\cos^2(\theta)}\cdot\cos(\theta)\,d\theta$$

$u = 2\theta$

$du = 2d\theta$

θ	u
$-\pi/2$	$-\pi$
$\pi/2$	π

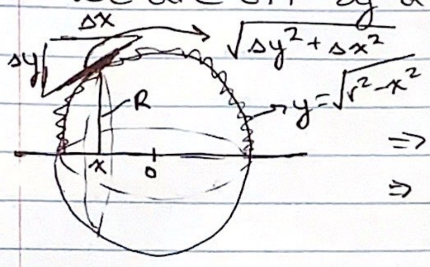
$$= 2\pi r^2\int_{-\pi/2}^{\pi/2}\cos^2(\theta)\,d\theta = \cancel{2}\pi r^2\int_{-\pi/2}^{\pi/2}\cancel{\tfrac{1}{2}}(1+\cos(2\theta))\,d\theta$$

$$= \pi r^2\int_{-\pi}^{\pi}(1+\cos(u))\cdot\tfrac{1}{2}du = \tfrac{1}{2}\pi r^2(u+\sin(u))\Big|_{-\pi}^{\pi} =$$ cont'd ↓

$$= \frac{1}{2}\pi r^2\left(\pi + \sin(\pi) - (-\pi + \sin(-\pi))\right)$$

$$= \frac{1}{2}\pi r^2\left(\pi + \cancel{} + \pi - \cancel{}\right) = \pi^2 r^2$$

The correct formula is $SA = 4\pi r^2$.
We are off by a factor of $\frac{4}{\pi}$. Why?



$$\Delta S = \sqrt{(\Delta x)^2 + (\Delta(y))^2}$$

$$\Rightarrow \Delta S = \sqrt{(\Delta x)^2\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)}$$

$$\Rightarrow \Delta S = \Delta x\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

$$\Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

The are contributed by this bit of arc is $2\pi R\,ds$

$$\Rightarrow SA = \int_a^b 2\pi R\,ds = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$R = y = \sqrt{r^2 - x^2} \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\Rightarrow SA = \int_{-r}^{r} 2\pi\sqrt{r^2 - x^2}\sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\,dx$$

$$= \int_{-r}^{r} 2\pi\sqrt{r^2 - x^2}\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx$$

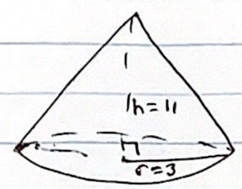
$$= \int_{-r}^{r} 2\pi\sqrt{(r^2 - x^2)\left(1 + \frac{x^2}{r^2 - x^2}\right)}\,dx$$

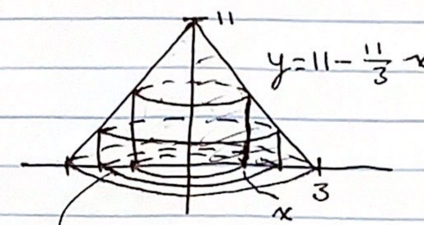
$$= \int_{-r}^{r} 2\pi\sqrt{r^2 - x^2 + x^2}\,dx$$

$$= \int_{-r}^{r} 2\pi r\,dx$$

$$= 2\pi r x\Big|_{-r}^{r} = 2\pi r^2 - (-2\pi r^2) = \underline{\underline{4\pi r^2}}$$

Right-circular cone with base radius 3 and height 11:





Cylindrical Shell Method:

$0 \le x \le 3$

$0 \le y \le 11$

$$V = \int_0^3 2\pi RH\,dx$$

$$= \int_0^3 2\pi xy\,dx$$

$$= \int_0^3 2\pi x\left(11 - \tfrac{11}{3}x\right)dx$$

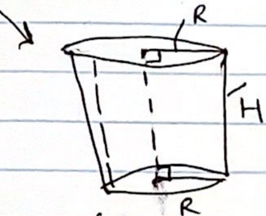
$$= 2\pi \int_0^3 \left(11x - \tfrac{11}{3}x^2\right)dx$$

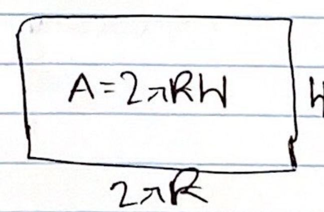
$$= 2\pi\left(\tfrac{11}{2}x^2 - \tfrac{11}{9}x^3\right)\Big|_0^3$$

$$= 2\pi\left(\tfrac{11\cdot 9}{2} - \tfrac{11}{9}\cdot 27 - 0\right)$$

$$= 2\pi\left(\tfrac{99}{2} - 33\right)$$

$$= 99\pi - 66\pi = 33\pi$$





$$y = 11 - \tfrac{11}{3}x \Rightarrow \frac{dy}{dx} = -\frac{11}{3}$$

$$SA = \int_0^3 2\pi R\,ds = \int_0^3 2\pi x\sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^3 2\pi x\sqrt{1 + \left(-\tfrac{11}{3}\right)^2}\,dx$$

$$= \int_0^3 2\pi x\sqrt{\tfrac{130}{9}}\,dx$$

$$= \frac{2\pi\sqrt{130}}{3}\int_0^3 x\,dx$$

$$= \frac{\cancel{2}\pi\sqrt{130}}{3}\cdot \frac{1}{\cancel{2}}x^2\Big|_0^3$$

$$= \frac{\pi\sqrt{130}}{3}\left(3^2 - 0\right) = 3\pi\sqrt{130}$$