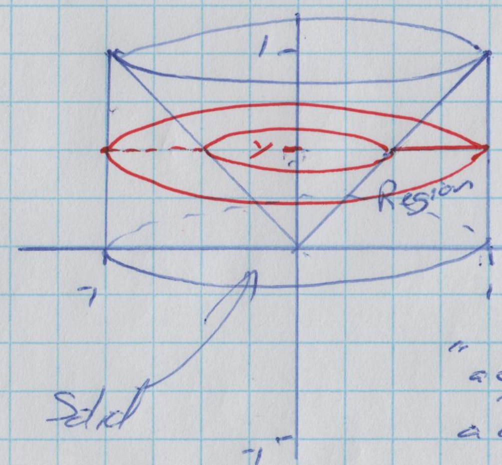
Volumes of Solids of Revolution

(ie of solids with circular symmetry about some line)

axis of revolution

These are solids created by revolving a region about some line (that doesn't pass inside the region).



outside radius R & inside radius r, so area $\pi R^2 - \pi r^2$

⇒ Suppose we revolve the region below $y=x$ and above $y=0$, for $0 \le x \le 1$, about the y-axis

"a cylinder with a cone cut out"

Disk or washer method: Use cross-sections perpendicular to the axis of revolution. You should use the variable whose axis is perpendicular to the cross-sections (ie parallel to the axis of revolution). In this case the variable is y, $R = 1-0 = 1$ & $r = x - 0 = x$, but $y = x$ so $r = y$.

2

$$\text{Volume} = \int_0^1 \pi (R^2 - r^2)\,dy = \int_0^1 \pi (1^2 - y^2)\,dy = \pi \int_0^1 (1 - y^2)\,dy$$

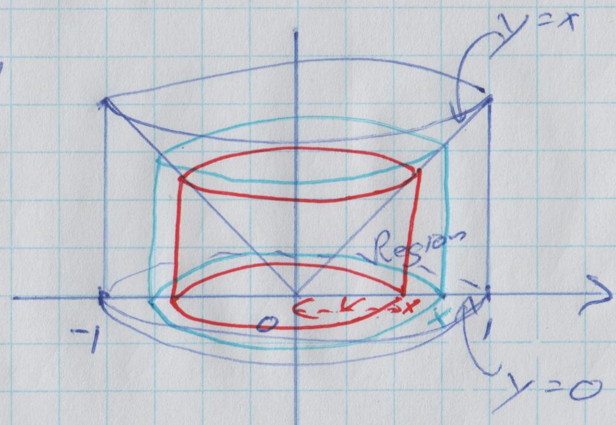
Since the region goes from $y=0$ to $y=1$.

$$= \pi \left(y - \frac{y^3}{3} \right) \Big|_0^1 = \pi \left(1 - \frac{1^3}{3}\right) - \cancel{\pi\left(0 - \frac{0^3}{3}\right)}$$

$$= \frac{2\pi}{3}$$

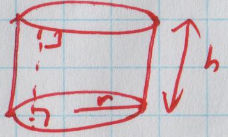
Alternatemethod: Cylindrical shell method

Use cross-sections that are nested cylinders with the ~~axis~~ axis of symmetry being the axis of revolution

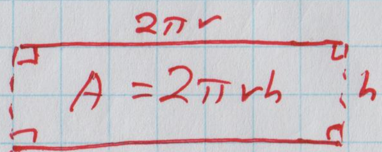


$0 \le x \le 1$

Area of cylinder



is ~~πr~~



In this case $r = x - 0 = x$

& $h = y - 0 = x$

Use the variable perpendicular to these cylindrical shells, i.e. x here,

i.e. the variable perpendicular to the axis of revolution

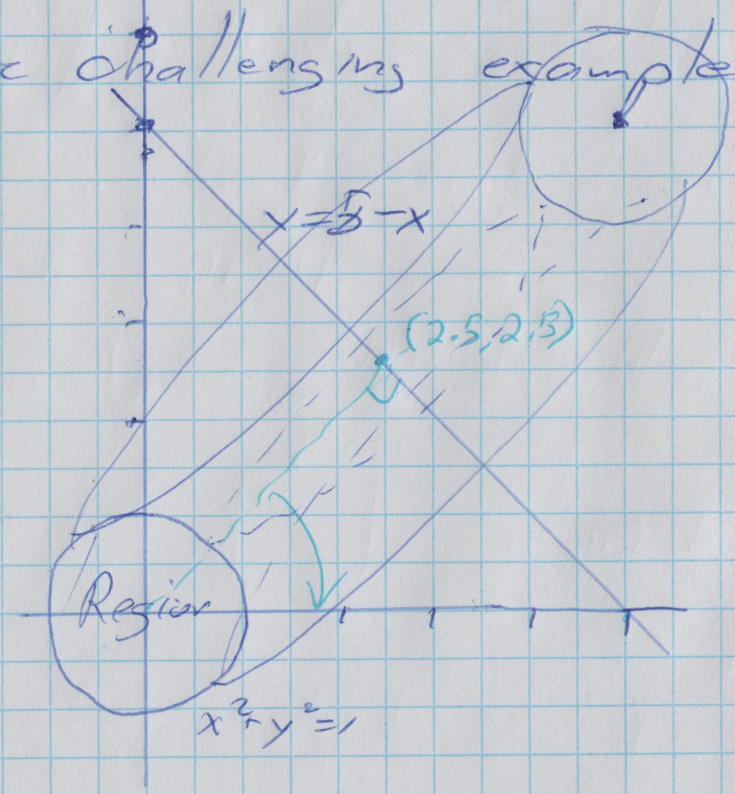
$$V = \int_0^1 \text{Area}\,dx = \int_0^1 2\pi r h\,dx = \int_0^1 2\pi x \cdot x\,dx$$

$0 \le x \le 1$ for the original region

$$= 2\pi \int_0^1 x^2\,dx = 2\pi \cdot \frac{x^3}{3}\Big|_0^1 = \frac{2\pi}{3}\cdot 1^3 - \cancel{\frac{2\pi}{3}\cdot 0^3}$$

$$= \frac{2\pi}{3}$$

A more challenging example:



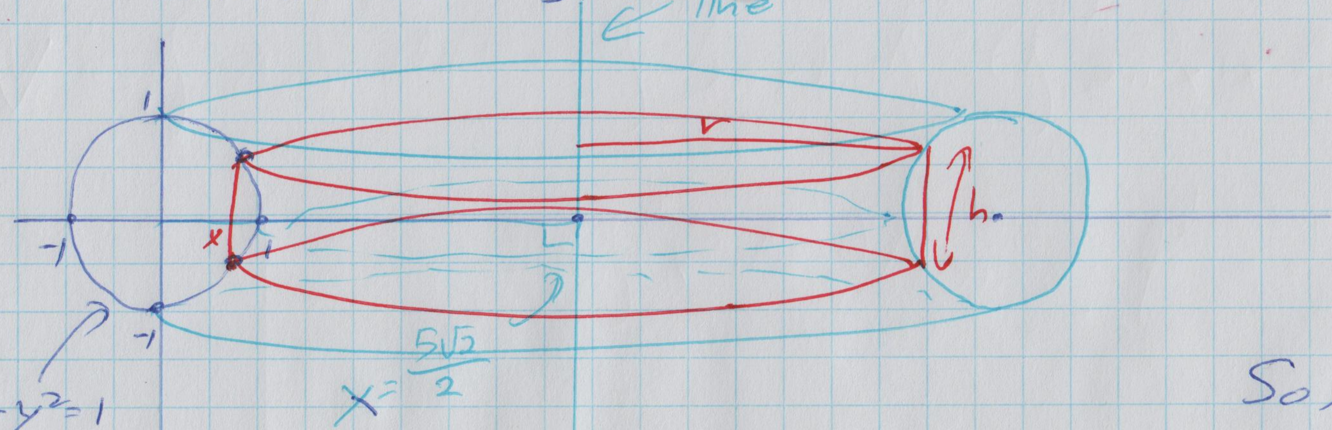
We want to revolve the unit disk centred at the origin about the line $y = 5 - x$

The line is neither horizontal nor vertical...

so we rotate this picture to make the line vertical

Rotate ~~the~~ about the origin ($\frac{\pi}{4}$ rad or 45° clockwise)

line



$x^2 + y^2 = 1$

$y = \pm\sqrt{1-x^2}$

$\sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{2 \cdot \frac{25}{4}} = \frac{5}{2}\sqrt{2}$

So, effectively, we're revolving the unit circle about the line $x = \frac{5}{2}\sqrt{2} = \frac{5}{\sqrt{2}}$

Cylindrical shell method:

Use x since the line is vertical & x-axis is perpendicular to the shells.

$r = \frac{5}{2}\sqrt{2} - x$

$h = \left(+\sqrt{1-x^2}\right) - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$

$$V = \int_{-1}^{1} 2\pi rh\,dx = \int_{-1}^{1} 2\pi\left(\frac{5}{2}\sqrt{2} - x\right) \cdot 2\sqrt{1-x^2}\,dx$$

5

$$= 4\pi \, \frac{5}{2}\sqrt{2} \int_{-1}^{1} \sqrt{1-x^2}\, dx + \left(-4\pi\right) \int_{-1}^{1} x\sqrt{1-x^2}\, dx$$

$-2\cdot 2$

x	θ
-1	$-\pi/2$
1	$\pi/2$

$x = \sin(\theta)$

$dx = \cos(\theta)\, d\theta$

$u = 1 - x^2$

$du = -2x\, dx$

x	u
-1	0
1	0

$$= 10\pi\sqrt{2} \int_{-\pi/2}^{\pi/2} \underbrace{\sqrt{1-\sin^2(\theta)}}_{\cos^2(\theta)} \cdot \cos(\theta)\, d\theta + \cancel{2\pi \int_{0}^{0} \sqrt{u}\, du} = 0$$

$$= 10\sqrt{2}\,\pi \int_{-\pi/2}^{\pi/2} \cos(\theta)\cdot\cos(\theta)\, d\theta = 10\sqrt{2}\,\pi \int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta$$

$\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$

$$= 10\sqrt{2}\,\pi \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1+\cos(2\theta)\right) d\theta$$

$w = 2\theta$

$dw = 2\, d\theta \Rightarrow \frac{1}{2} dw = d\theta$

θ	w
$-\pi/2$	$-\pi$
$\pi/2$	π

$$= \frac{10\sqrt{2}\,\pi}{2} \int_{-\pi}^{\pi} \left(1+\cos(w)\right)\cdot \frac{1}{2}\, dw = \frac{5}{2}\sqrt{2}\,\pi \left(w + \sin(w)\right)\Big|_{-\pi}^{\pi}$$

$$= \frac{5}{2}\sqrt{2}\,\pi \cdot \left[(\pi+0) - (-\pi+0)\right] = \frac{5}{\cancel{2}}\sqrt{2}\,\pi \cdot \cancel{2}\pi = 5\sqrt{2}\cdot\pi^2$$