Improper Integrals I

"Proper" definite integrals looks like $\int_a^b f(x)dx$, where a & b are real numbers

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \underset{G}{?}$$

change it $\int_0^{\infty} xe^{-x^2}dx$ → we can't throw around ∞ like a number

$$= \lim_{a\to\infty}\int_0^a xe^{-x^2}dx \longrightarrow w=-x^2,\ dw=-2xdx,\ -\tfrac{1}{2}dw = xdx$$

$$= \lim_{a\to\infty}\int_{x=0}^{x=a} e^{w}\left(-\tfrac{1}{2}\right)dw$$

$$= \lim_{a\to\infty}\left(-\tfrac{1}{2}\right)\int_{x=0}^{x=a} e^{w}\,dw$$

$$= \lim_{a\to\infty}\left(-\tfrac{1}{2}\right)e^{w}\Big|_{x=0}^{x=a}$$

$$= \lim_{a\to\infty} -\tfrac{1}{2}e^{-x^2}\Big|_0^a$$

$$= \lim_{a\to\infty}\left(-\tfrac{1}{2}e^{-a^2} + \tfrac{1}{2}e^{-0^2}\right)$$

$$= \lim_{a\to\infty}\left(-\tfrac{1}{2}e^{-a^2} + \tfrac{1}{2}\cdot 1\right)$$

$$= \lim_{a\to\infty}\tfrac{1}{2}\Big(1-\underbrace{e^{\overset{\to -\infty}{-a^2}}}_{0}\Big)$$

$$= \tfrac{1}{2}[1-0] \quad = \tfrac{1}{2}$$

ex: $\int_0^1 \frac{1}{\sqrt{x}}\,dx \longrightarrow \frac{1}{\sqrt{x}}$ has a vertical asymptote as $x\to 0^+$

$$= \lim_{b\to 0^+}\int_b^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{b\to 0^+}\int_b^1 x^{-1/2}\,dx$$

$$= \lim_{b\to 0^+} \left. \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right|_b^1$$

$$= \lim_{b\to 0^+} \left. \frac{x^{1/2}}{1/2} \right|_b^1$$

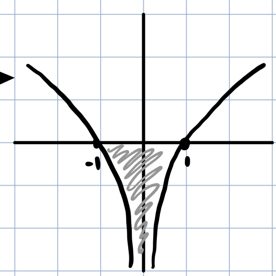
$$= \lim_{b\to 0^+} \left. 2\sqrt{x} \right|_b^1$$

$$= \lim_{b\to 0^+} \left(2\sqrt{1} - 2\sqrt{b}\right)$$

$$= \lim_{b\to 0^+} \left(2 - 2\sqrt{b}\right) \to \sqrt{0}$$

$$= \lim_{b\to 0^+} (2-0) \quad = 2$$

ex: $\int_{-1}^{1} \ln(|x|)\,dx$ — graph of $\ln(|x|)$ →

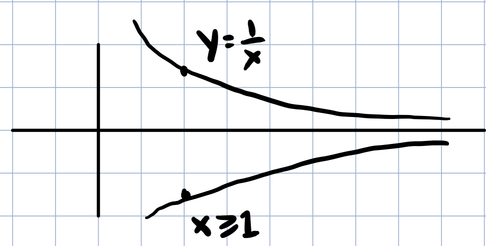


$$= \int_{-1}^{0} \ln(|x|)\,dx + \int_{0}^{1} \ln(|x|)\,dx$$

$$= \lim_{a\to 0^-} \int_{-1}^{a} \ln(|x|)\,dx + \lim_{b\to 0^+} \int_{b}^{1} \ln(|x|)\,dx$$

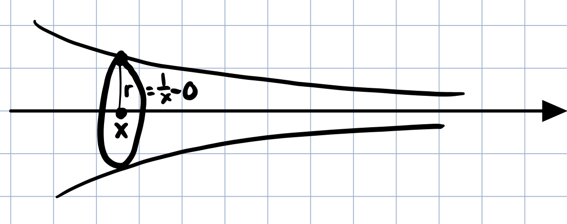
$$= \lim_{a\to 0^-} \int_{-1}^{a} \ln(-x)\,dx + \lim_{b\to 0^+} \int_{b}^{1} \ln(x)\,dx$$ → use dummy product

What is the volume of "Gabriel's Trumpet" ie:



[Revolve the region below $y = \frac{1}{x}$ and above $y=0$ about the x-axis $(x \geq 1)$]

$$V = \int_1^{\infty} \pi \left(\frac{1}{x}\right)^2 dx$$



$$= \lim_{a\to\infty} \int_1^{\infty} \frac{\pi}{x^2}\,dx$$

$$= \lim_{a\to\infty} \int_1^{a} \pi x^{-2}\,dx$$

$$= \lim_{a\to\infty} \frac{\pi x^{-1}}{-1}\Bigg|_1^a$$

$$= \lim_{a\to\infty} \frac{-\pi}{x}\Bigg|_1^a$$

$$= \lim_{a\to\infty} \left[\frac{-\pi}{a} - \left(-\frac{\pi}{1}\right)\right.$$

$$= \lim_{a\to\infty} \left(-\frac{\pi}{a} + \pi\right)$$

$$= 0 + \pi \quad = \pi$$