Integrating Rational Functions III

What to do with more in the denominater:

$$\int \frac{x^2+x+1}{(ax^2+bx+c)^3(dx^2+ex+f)(x-g)^2}\,dx \longrightarrow \text{assume these are irreducible quadratic}$$

$$=\int \frac{Ax+B}{(ax^2+bx+c)^3}dx+\int\frac{Cx+D}{(ax^2+bx+c)^2}dx+\int\frac{Ex+F}{ax^2+bx+c}dx+\int\frac{Gx+H}{dx^2+ex+f}dx+\int\frac{I}{(x-g)^2}dx+\int\frac{J}{x-g}dx$$

- Break it up according to a complete factorization of the denominator and step down from the biggest power of each factor

To do any of this you need a fully factored denominator... and factoring polynomials is hard.

X^5+X^3+x+15 factor this??

We can factor quadratics (or tell if they're irreducible) by using the quadratic formula.

ex: $3x^2+2x+4=0$ $\qquad \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$=\frac{-2\pm\sqrt{2^2-4(3)(4)}}{2\cdot 3}$$

$$=\frac{-2\pm\sqrt{56}}{6} \quad \text{so this will factor as } 3\left(x-\frac{-2+\sqrt{56}}{6}\right)\left(x+\frac{-2-\sqrt{56}}{6}\right)$$

There are cubic & quadratic formulas that can be used for 3^{rd} & 4^{th} degree, but after that there are no general factorization formulas.

ex: $\int \frac{1}{x^4-1}dx$

$$=\int\frac{1}{(x^2)^2-1}dx$$

$$=\int\frac{1}{(x^2-1)(x^2+1)}dx$$

$$=\int\frac{1}{(x-1)(x+1)(x^2+1)}dx$$

$$\frac{1}{(x-1)(x+1)(x^2+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

$$=\frac{A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x-1)(x+1)}{(x-1)(x+1)(x^2+1)}$$

$$=\frac{A(x^3+x^2+x+1)+B(x^3-x^2+x-1)+(Cx+D)(x^2-1)}{(x-1)(x+1)(x^2+1)}$$

$$=\frac{Ax^3+Ax^2+Ax+A+Bx^3-Bx^2+Bx-B+Cx^3+Dx^2-Cx-D}{(x-1)(x+1)(x^2+1)}$$

$$=\frac{(A+B+C)x^3+(A-B+D)x^2+(A+B-C)x+(A-B-D)}{(x-1)(x+1)(x^2+1)}$$

$$\frac{1}{(x-1)(x+1)(x^2+1)}$$

$$\frac{0x^3+0x^2+0x+1}{(x-1)(x+1)(x^2+1)}$$

Using linear alegebra

1) $A+B+C=0$
2) $A-B+D=0$
3) $A+B-C=0$
4) $A-B-D=1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 \end{array}\right] \xrightarrow[R_4-R_1]{R_2-R_1,\ R_3-R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_4-R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right] \xrightarrow{R_2+\frac{1}{2}R_4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right] \xrightarrow[R_1+\frac{1}{2}R_3]{R_2-\frac{1}{2}R_3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right] \xrightarrow{R_1+\frac{1}{2}R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & -2 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right] \xrightarrow[-\frac{1}{2}R_3,\ -\frac{1}{2}R_4]{-\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

$A=\frac{1}{4}$
$B=-\frac{1}{4}$
$C=0$
$D=-\frac{1}{2}$

Solving using subsitution

$2)-4) = 2D = -1$
$D = -\frac{1}{2}$

$1)-3) = 2C = 0$
$C = 0$

1) $A+B=0$
2) $A-B=\frac{1}{2}$

$1)+2) = 2A = \frac{1}{2}$
$A = \frac{1}{4}$

$1)-2) = 2B = -\frac{1}{2}$
$B = -\frac{1}{4}$

$$= \int \frac{1/4}{x-1}\,dx + \int \frac{-1/4}{x+1}\,dx + \int \frac{0x-\frac{1}{2}}{x^2-1}\,dx$$

$$= \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2-1}\,dx$$

$u = x-1$
$du = dx$

$w = x+1$
$dw = dx$

$$= \frac{1}{4}\int \frac{1}{u}\,du - \frac{1}{4}\int \frac{1}{w}\,dw - \frac{1}{2}\arctan(x)$$

$$= \frac{1}{4}\ln(u) - \frac{1}{4}\ln(w) - \frac{1}{2}\arctan(x) + C$$

$$= \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x+1) - \frac{1}{2}\arctan(x) + C$$