Integrating Rational Functions II

-How to break them up... using more algebra

1) We know what to do if $\int \frac{p(x)}{q(x)}\,dx$ and $p(x)$ has degree $\geq$ the degree of $q(x)$

We divid $q(x)$ into $p(x)$, so $p(x) = s(x)q(x) + r(x)$ (where $r(x)$ is guaranteed to have degree $\leq$ degree of $q(x)$)

$$\int \frac{p(x)}{q(x)}\,dx = \int \left[\frac{r(x)}{q(x)} + s(x)\right] dx$$

2) We know what to do if we have something of the form $\int \frac{ax+b}{(cx+d)^n}\,dx$ or $\int \frac{ax+b}{(cx^2+dx+e)^k}\,dx$

These can be handled by a comlpination of algebra (eg: completing the square) substitution and/or trig substitution

Warm up:

$$\int \frac{x-1}{3x^2+2x+4}\,dx \longrightarrow \text{try } u = 3x^2+2x+4,\ du = (6x+2)\,dx$$
$$= \tfrac{1}{6}du = (x+\tfrac{1}{3})\,dx$$

$$= \int \frac{x+\frac{1}{3}-\frac{1}{3}-1}{3x^2+2x+4}\,dx$$

$$= \int \frac{x+\frac{1}{3}}{3x^2+2x+4}\,dx - \int \frac{\frac{4}{3}}{3x^2+2x+4}\,dx$$

$$= \int \frac{1}{u}\cdot\frac{1}{6}\,du - \frac{4}{3}\int \frac{1}{3x^2+2x+4}\,dx \longrightarrow \text{complete the square}$$

$$3x^2+2x+4 = 3\left(x^2+\tfrac{2}{3}x+\tfrac{4}{3}\right) = 3\left(\left(x+\tfrac{1}{3}\right)^2-\tfrac{1}{9}+\tfrac{4}{3}\right) = 3\left(\left(x+\tfrac{1}{3}\right)^2+\tfrac{11}{9}\right)$$

$$= \frac{1}{6}\ln(u) - \frac{4}{3}\int \frac{1}{3\left(\left(x+\frac{1}{3}\right)^2+\frac{11}{9}\right)}\,dx$$

$$= \frac{1}{6}\ln(u) - \frac{4}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\frac{11}{9}}\,dx \longrightarrow \text{plug } 3x^2+2x+4 \text{ in for } u$$
$$\longrightarrow \text{sub } w = x+\tfrac{1}{3},\ dw = dx$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{9}\int \frac{1}{(w)^2+\frac{11}{9}}\,dw \longrightarrow w = \sqrt{\tfrac{11}{9}}\tan(\theta),\ dw = \sqrt{\tfrac{11}{9}}\sec^2(\theta)\,d\theta$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{9}\int \frac{1}{\left(\sqrt{\frac{11}{9}}\tan\theta\right)^2+\frac{11}{9}}\cdot\sqrt{\tfrac{11}{9}}\sec^2(\theta)\,d\theta$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{9}\int \frac{\sqrt{\frac{11}{9}}\sec^2(\theta)}{\frac{11}{9}(\tan^2\theta+1)}\,d\theta$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{9}\cdot\frac{1}{\sqrt{11/9}}\int \frac{\cancel{\sec^2(\theta)}}{\cancel{\sec^2(\theta)}}\,d\theta$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{3\sqrt{11}}\cdot\theta + C \longrightarrow w=\sqrt{\tfrac{11}{9}}\tan\theta$$

$$\frac{w}{\sqrt{\frac{11}{9}}} = \tan\theta$$

$$\theta = \arctan\left(\frac{w}{\sqrt{\frac{11}{9}}}\right)$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{3\sqrt{11}}\cdot\arctan\left(\frac{w}{\sqrt{\frac{11}{9}}}\right) + C \longrightarrow w = x+\frac{1}{3}$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{3\sqrt{11}}\cdot\arctan\left(\frac{x+1/3}{\sqrt{11}/3}\right) + C$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{3\sqrt{11}}\cdot\arctan\left(\frac{3x+1}{\sqrt{11}}\right) + C$$

ex: $\int \frac{x^2+3x+41}{(x^2+1)(x+1)}\,dx \longrightarrow$ Partial fractions $\frac{x^2+3x+41}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$, where A,B,C are some constants

$$= \frac{(Ax+B)(x+1)+C(x^2+1)}{(x^2+1)(x+1)}$$

$$= \frac{Ax^2+Ax+Bx+B+Cx^2+C}{(x^2+1)(x+1)}$$

$$= \frac{(A+C)x^2+(A+B)x+(B+C)}{(x^2+1)(x+1)}$$

1) $A+C=1$
2) $A+B=3$
3) $B+C=41$

$$= \int \frac{\frac{-37}{2}x+\frac{43}{2}}{x^2+1}\,dx + \int \frac{\frac{39}{2}}{x+1}\,dx$$

Using linear algebra: $1A+0B+1C=1$
$1A+1B+0C=3$
$0A+1B+1C=41$

$$\left[\begin{array}{ccc|c}1&0&1&1\\1&1&0&3\\0&1&1&41\end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&2\\0&1&1&41\end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&2\\0&0&2&39\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&2\\0&0&1&\frac{39}{2}\end{array}\right] \xrightarrow[R_1-R_3]{R_2+R_3} \left[\begin{array}{ccc|c}1&0&0&\frac{-37}{2}\\0&1&0&\frac{43}{2}\\0&0&1&\frac{39}{2}\end{array}\right] \longrightarrow \begin{array}{l}A=\frac{-37}{2}\\B=\frac{43}{2}\\C=\frac{39}{2}\end{array}$$

1 way of solving
1) $A=1-C$
2) $1-C+B=3$
$B=C+2$
3) $C+2+C=41$
$2C=39$
$C=39/2$
2) $B=\frac{39}{2}+2$
$B=43/2$
1) $A=1-\frac{39}{2}$
$A=\frac{-37}{2}$