Integrating Rational Functions I

A rational function is a ratio of polynomials (fractions)

ex $\int \frac{7x+4}{3x+2}\,dx$

$= \int \frac{7x}{3x+2}\,dx + \int \frac{4}{3x+2}\,dx \longrightarrow u = 3x+2,\ du = 3dx,\ dx = \frac{1}{3}du$

$x = \frac{u-2}{3}$

$= \int \frac{7\left(\frac{u-2}{3}\right)}{u} \cdot \frac{1}{3}\,dx + \int \frac{4}{u} \cdot \frac{1}{3}\,dx$

$= \frac{7}{9} \int \frac{u-2}{u}\,du + \frac{4}{3} \int \frac{1}{u}\,du$

$= \frac{7}{9} \int \left(1 - \frac{2}{u}\right)du + \frac{4}{3} \int \frac{1}{u}\,du$

$= \frac{7}{9}\left(u - 2\ln(u)\right) + \frac{4}{3}(\ln(u)) + C$

$= \frac{7}{9}u - \frac{14}{9}\ln(u) + \frac{12}{9}\ln(u) + C$

$= \frac{7}{9}u - \frac{2}{9}\ln(u) + C$

$= \frac{7}{9}(3x+2) - \frac{2}{9}(3x+2) + C$

ex $\int \frac{3x+43}{x^2+8x+34}\,dx \longrightarrow$ complete the square $\quad x^2+8x+34 = (x+4)^2+18$

b/c $(x+4)^2 = x^2+8x+16$

$= \int \frac{3x+43}{(x+4)^2+18}\,dx \longrightarrow u = x+4,\ du = 1$

$= \int \frac{3x+43}{u^2+18}\,du \longrightarrow x = u-4$

$= \int \frac{3(u-4)+43}{u^2+18}\,du$

$= \int \frac{3u+31}{u^2+18}\,du$

$= \int \frac{3u}{u^2+18}\,du + \int \frac{31}{u^2+18}\,du$

$= 3\int \frac{u}{u^2+18}\,du + 31\int \frac{1}{u^2+18}\,du$

$\rightarrow w = u^2+18,\ dw = 2u\,du,\ \frac{1}{2}dw = u\,du$

$\rightarrow$ trig subsitution , $u = \sqrt{18}\tan\theta$

$du = \sqrt{18}\,cec^2\theta\, d\theta$

$= \frac{3}{2}\int \frac{1}{w}\,dw + 31\int \frac{1}{\underbrace{(\sqrt{18}\tan\theta)^2}_{18\tan^2\theta}+18}\ \sqrt{18}\sec^2\theta\, du$

$= \frac{3}{2}\ln(w) + 31\cdot\sqrt{18}\int \frac{1}{18(\tan^2\theta+1)}\ \sec^2\theta\, d\theta \longrightarrow \tan^2\theta + 1 = \sec^2\theta$

$= \frac{3}{2}\ln(w) + \frac{31\cdot\sqrt{18}}{18}\int 1\, d\theta$

$= \frac{3}{2}\ln(w) + \frac{31}{\sqrt{18}}\,\theta + C$

$\rightarrow u = \sqrt{18}\tan\theta$

$\frac{u}{\sqrt{18}} = \tan\theta$

$\theta = \arctan\left(\frac{u}{\sqrt{18}}\right)$

$= \frac{3}{2}\ln(u^2+18) + \frac{31}{\sqrt{18}}\left(\arctan\left(\frac{u}{\sqrt{18}}\right)\right) + C$

$= \frac{3}{2}\ln\left((x+4)^2+18\right) + \frac{31}{\sqrt{18}}\arctan\left(\frac{x+4}{\sqrt{18}}\right) + C$

ex: $\int \frac{x^3+x^2+2x+1}{x^2+4x+3}\,dx$

[Rational functions are easyer to integrate if the degree of the numerator is less then the degree of the denometer.]

$\rightarrow$ divide denominators into numarator

$$
\begin{array}{r|l}
 & x \ \ -3 \\
x^2+4x+3 & x^3+x^2+2x+1 \\
 & -(x^3+4x^2+3x) \downarrow \\
 & \quad -3x^2-x\ +1 \\
 & \quad -(-3x^2-12x-9) \\
 & \qquad 11x+10
\end{array}
\quad \Bigg\} \quad
\begin{aligned}
&x^3+x^2+2x+1 \\
&= (x-3)(x^2+4x+3) + (11x+10)
\end{aligned}
$$

$= \int \frac{(x-3)(x^2+4x+3)+(11x+10)}{x^2+4x+3}$

$= \int (x-3)\,dx + \int \frac{11x+10}{x^2+4x+3}$