Trogonometric Substitutions II

So far, see $\sqrt{1-x^2}$, try $x=\sin\theta$, so $dx=\cos\theta\, d\theta$

see $\sqrt{1+x^2}$, try $x=\tan\theta$, so $dx=\sec^2\theta\, d\theta$

see $\sqrt{x^2-1}$, try $x=\sec\theta$, so $dx=\sec\theta\tan\theta$

see $\sqrt{-1-x^2}$, try not to panic (wait till third year)

ex $\int (x^2+4)^{3/2}\,dx \longrightarrow x=2\tan\theta$, so $dx=2\sec^2\theta\, d \rightarrow$ [$2=\sqrt{4}$ why we picked 2 infront of $\tan\theta$

x^2+4, $x=2u$, $(2u)^2+4$, $4u^2+4$, $4(u^2+1)$]

$=\int (2^2\tan^2\theta+4)^{3/2}\, 2\sec^2\theta\, d\theta$

$=2\int (4\tan^2\theta+4)^{3/2}\sec^2\theta\, d\theta$

$=2\int (4(\tan^2\theta+1))^{3/2}\sec^2\theta\, d\theta$

$=2\int 8(\sec^2(\theta))^{3/2}\sec^2\theta\, d\theta$

$=16\int \sec^3\theta\sec^2\theta\, d\theta$

$=16\int \sec^5\theta\, d\theta \longrightarrow$ reduction formulas

$=16\left[\frac{1}{5-1}\tan\theta\sec^{5-2}\theta+\frac{5-2}{5-1}\int\sec^{5-2}\theta\, d\theta\right]$

$=16\left[\frac{1}{4}\tan\theta\sec^3\theta+\frac{3}{4}\int\sec^3\theta\, d\theta\right]$

$=4\tan\theta\sec^3\theta+12\int\sec^3\theta\, d\theta \longrightarrow$ reduction formula again

$=4\tan\theta\sec^3\theta+12\left[\frac{1}{3-1}\tan\theta\sec^{3-2}\theta+\frac{3-2}{3-1}\int\sec^{3-2}\theta\, d\theta\right]$

$=4\tan\theta\sec^3\theta+12\left[\frac{1}{2}\tan\theta\sec\theta+\frac{1}{2}\int\underline{\sec\theta\, d\theta}\right]$ → memerize

$=4\tan\theta\sec^3\theta+6\tan\theta\sec\theta+6\ln(\sec\theta+\tan\theta)+C$

$=4\left(\frac{x}{2}\right)\left(\sqrt{1+\frac{x^2}{4}}\right)^3+6\left(\frac{x}{2}\right)\left(\sqrt{1+\frac{x^2}{4}}\right)+6\ln\left(\sqrt{1+\frac{x^2}{4}}+\frac{x}{2}\right)+C$

$\xrightarrow{\text{solve for } \tan\theta}$ $x=2\tan\theta$, $\frac{x}{2}=\tan\theta$

$\xrightarrow{\text{solve for } \sec\theta}$ $1+\tan^2\theta=\sec^2\theta$

$\sqrt{1+\tan^2\theta}=\sec\theta$

$\sec\theta=\sqrt{1+\left(\frac{x}{2}\right)^2}$

$=\sqrt{1+\frac{x^2}{4}}$

> Note: Some functions do **NOT** have antiderivatives
> ex $\frac{1}{\sqrt{2\pi}}\int e^{-x^2/2}\,dx$

most like $\sqrt{x^2-1}$

ex: $\int \frac{1}{\sqrt{9x^2-4}}\,dx \rightarrow x = \frac{2}{3}\sec\theta$, so $dx = \frac{2}{3}\sec\theta\tan\theta\,d\theta$

$= \int \frac{1}{\sqrt{9(\frac{2}{3}\sec\theta)^2-4}} \cdot \frac{2}{3}\sec\theta\tan\theta\,d\theta$

$= \int \frac{1}{\sqrt{9\cdot\frac{4}{9}\sec^2\theta-4}} \cdot \frac{2}{3}\sec\theta\tan\theta\,d\theta$

$= \frac{2}{3}\int \frac{1}{\sqrt{4(\sec^2\theta-1)}} \cdot \sec\theta\tan\theta\,d\theta$

$= \frac{\cancel{2}}{3}\int \frac{1}{\cancel{2}\sqrt{\tan^2\theta}} \cdot \sec\theta\tan\theta\,d\theta$

$= \frac{1}{3}\int \frac{\sec(\theta)\cancel{\tan(\theta)}}{\cancel{\tan\theta}}$

$= \frac{1}{3}\int \sec\theta\,d\theta$

$= \frac{1}{3}\ln(\sec\theta+\tan\theta)+C$

$= \frac{1}{3}\ln\left(\frac{3x}{2}+\sqrt{\frac{9x^2-1}{4}}\right)+C$

$x = \frac{2}{3}\sec\theta$

$\Rightarrow \sec\theta = \frac{3x}{2}$

$1+\tan^2\theta = \sec^2\theta$

$\tan\theta = \sqrt{\sec^2(\theta)-1}$

$= \sqrt{\frac{9x^2}{4}-1}$

If you see...

$\sqrt{a^2-b^2x^2}$, $x = \frac{a}{b}\sin(\theta)$

$\sqrt{a^2+b^2x^2}$, $x = \frac{a}{b}\tan(\theta)$

$\sqrt{b^2x^2-a^2}$, $x = \frac{a}{b}\sec(\theta)$