Lecture 7 Feb 1st, 2022

Recall: see $\sqrt{1-x^2} \Rightarrow$ try $x = \sin(\theta)$, $dx = \cos(\theta)d\theta$

" $\sqrt{1+x^2} \Rightarrow$ try $x = \tan(\theta)$, $dx = \sec^2(\theta)d\theta$

" $\sqrt{x^2-1} \Rightarrow$ try $x = \sec(\theta)$, $dx = \sec(\theta)\tan(\theta)d\theta$

ex/ $\int (x^2+4)^{3/2} dx$ $\qquad x = 2\tan(\theta)$, $dx = 2\sec^2(\theta)d\theta$

$= \int (2^2\tan^2(\theta) + 4)^{3/2} 2\sec^2(\theta)d\theta$ $\qquad \tan(\theta) = \frac{x}{2}$

$= 2\int (4\tan^2(\theta)+4)^{3/2} \sec^2(\theta)d\theta$ $\qquad 1+\tan^2(\theta) = \sec^2(\theta)$

$= 2\int (4(\tan^2(\theta)+1))^{3/2} \sec^2(\theta)d\theta$ $\qquad \Rightarrow \sec(\theta) = \sqrt{1+\tan^2(\theta)}$

$\qquad = \sqrt{1+(\frac{x}{2})^2}$

$= 2\int 8(\tan^2(\theta)+1)^{3/2} \sec^2(\theta)d\theta$ $\qquad = \sqrt{1+\frac{x^2}{4}}$

$= 16\int (\sec^2(\theta))^{3/2} \sec^2(\theta)d\theta$

$= 16\int \sec^5(\theta)d\theta$

$= 16\left[\frac{1}{5-1}\tan(\theta)\sec^{5-2}(\theta) + \frac{5-2}{5-1}\int \sec^{5-2}(\theta)d\theta\right]$

$= 16\left[\frac{1}{4}\tan(\theta)\sec^3(\theta) + \frac{3}{4}\int \sec^3(\theta)d\theta\right]$

$= 4\tan(\theta)\sec^3(\theta) + 12\int \sec^3(\theta)d\theta$

$= 4\tan(\theta)\sec^3(\theta) + 12\left[\frac{1}{3-1}\tan(\theta)\sec^{3-2}(\theta) + \frac{3-2}{3-1}\int \sec^{3-2}(\theta)d\theta\right]$

$= 4\tan(\theta)\sec^3(\theta) + 12\left[\frac{1}{2}\tan(\theta)\sec(\theta) + \frac{1}{2}\int \sec(\theta)d\theta\right]$

$= 4\tan(\theta)\sec^3(\theta) + 6\tan(\theta)\sec(\theta) + 6\ln(\sec(\theta)+\tan(\theta)) + C$

$= 4\left(\frac{x}{2}\right)\left(\sqrt{1+\frac{x^2}{4}}\right)^3 + 6\left(\frac{x}{2}\right)\left(\sqrt{1+\frac{x^2}{4}}\right) + 6\ln\left(\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right) + C$

$= 2x\left(1+\frac{x^2}{4}\right)^{3/2} + 3x\sqrt{1+\frac{x^2}{4}} + 6\ln\left(\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right) + C$

Note: some functions have no antiderivatives.

ex/ $\frac{1}{\sqrt{2\pi}}\int e^{-x^2/2}\,dx$

ex/ $\int \frac{1}{\sqrt{9x^2-4}}\,dx$

$x = \frac{2}{3}\sec(\theta)$

$dx = \frac{2}{3}\sec(\theta)\tan(\theta)\,d\theta$

$\sec(\theta) = \frac{3}{2}x$

$\tan(\theta) = \sqrt{\sec^2(\theta)-1}$

$= \sqrt{(\frac{3}{2}x)^2-1}$

$= \sqrt{\frac{9}{4}x^2-1}$

$= \int \frac{1}{\sqrt{9(\frac{2}{3}\sec(\theta))^2-4}} \cdot \frac{2}{3}\sec(\theta)\tan(\theta)\,d\theta$

$= \int \frac{1}{\sqrt{9(\frac{4}{9}\sec^2(\theta))-4}} \cdot \frac{2}{3}\sec(\theta)\tan(\theta)\,d\theta$

$= \frac{2}{3}\int \frac{1}{\sqrt{4(\sec^2(\theta)-1)}} \sec(\theta)\tan(\theta)\,d\theta$

$= \frac{\cancel{2}}{3}\int \frac{1}{\cancel{2}\sqrt{\tan^2(\theta)}} \sec(\theta)\tan(\theta)\,d\theta$

$= \frac{1}{3}\int \frac{\sec(\theta)\cancel{\tan(\theta)}}{\cancel{\tan(\theta)}}\,d\theta$

$= \frac{1}{3}\int \sec(\theta)\,d\theta$

$= \frac{1}{3}\ln(\sec(\theta)+\tan(\theta)) + C$

$= \frac{1}{3}\ln\left(\frac{3}{2}x + \sqrt{\frac{9}{4}x^2-1}\right) + C$

* If you see $\sqrt{a^2-b^2x^2}$, let $x = \frac{a}{b}\sin(\theta)$

" $\sqrt{a^2+b^2x^2}$, let $x = \frac{a}{b}\tan(\theta)$

" $\sqrt{b^2x^2-a^2}$, let $x = \frac{a}{b}\sec(\theta)$

How to handle $\int \frac{1}{\sqrt{\text{polynomial}}}\,dx$

ex/ $\displaystyle\int \frac{1}{\sqrt{4x^2+2x+1}}dx$

$\cancel{x^2+\frac{1}{2}x+\frac{1}{4}}$
$\cancel{=(x^2+\frac{1}{2}x+\frac{1}{4}-\frac{1}{4})+\frac{1}{4}}$
$\cancel{=(x^2+\frac{1}{2}x+\frac{1}{4})-\frac{1}{4}+\frac{1}{4}}$

$4x^2+2x+1=4(x^2+\frac{1}{2}x+\frac{1}{4})$

$\Rightarrow =\frac{1}{2}\displaystyle\int \frac{1}{\sqrt{x^2+\frac{1}{2}x+\frac{1}{4}}}dx$

$x^2+\frac{1}{2}x+\frac{1}{4}=(x+\frac{1}{4})^2-\frac{(\frac{1}{2})^2}{2}+\frac{1}{4}=(x+\frac{1}{4})^2+\frac{1}{8}$

$\Rightarrow =\frac{1}{2}\displaystyle\int \frac{1}{\sqrt{(x+\frac{1}{4})^2+\frac{1}{8}}}$ $\qquad u=x+\frac{1}{4},\ du=dx$

$\Rightarrow =\frac{1}{2}\displaystyle\int \frac{1}{\sqrt{u^2+\frac{1}{8}}}du$ $\qquad u=\frac{1}{\sqrt{8}}\tan(\theta),\ du=\frac{1}{\sqrt{8}}\sec^2(\theta)d\theta$

$\Downarrow$

$\tan(\theta)=\sqrt{8}\cdot u$

$\sec(\theta)=\sqrt{1+\tan^2(\theta)}$
$=\sqrt{1+(\sqrt{8}u)^2}$
$=\sqrt{1+8u^2}$

$\Rightarrow =\frac{1}{2}\displaystyle\int \frac{1}{\sqrt{(\frac{1}{\sqrt{8}}\tan(\theta))^2+\frac{1}{8}}}\cdot\frac{1}{\sqrt{8}}\sec^2(\theta)d\theta$

$\Rightarrow =\frac{1}{2}\displaystyle\int \frac{1}{\sqrt{\frac{1}{8}\tan^2(\theta)+\frac{1}{8}}}\cdot\frac{1}{\sqrt{8}}\sec^2(\theta)d\theta$

$=\frac{1}{2\sqrt{8}}\displaystyle\int \frac{\sec^2(\theta)}{\sqrt{\frac{1}{8}(\tan^2(\theta)+1)}}d\theta$

$=\frac{1}{2\sqrt{8}\sqrt{\frac{1}{8}}}\displaystyle\int \frac{\sec^2(\theta)}{\sqrt{\tan^2(\theta)+1}}d\theta$

$=\frac{\cancel{\sqrt{8}}}{2\cancel{\sqrt{8}}}\displaystyle\int \frac{\sec^2(\theta)}{\sec(\theta)}d\theta$

$=\frac{1}{2\cancel{\sqrt{8}}}\displaystyle\int \sec(\theta)d\theta$

$=\frac{1}{2\cancel{\sqrt{8}}}\ln(\sec(\theta)+\tan(\theta))+C$

$=\ln(\sqrt{1+8u^2}+\sqrt{8}\cdot u)+C$

$=\ln(\sqrt{1+8(x+\frac{1}{4})^2}+\sqrt{8}\cdot(x+\frac{1}{4}))+C$

Alternative

Recall: $\cosh(x) = \frac{e^x + e^{-x}}{2}$ $\quad \cosh^2(x) = \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4}$

$= \frac{e^{2x} + e^{-2x} + 2}{4}$

$\sinh(x) = \frac{e^x - e^{-x}}{2}$

$\sinh^2(x) = \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}$

$= \frac{e^{2x} + e^{-2x} - 2}{4}$

$\cosh^2(x) - \sinh^2(x) = 1$

$\frac{d}{dx}\cosh(x) = \sinh(x)$, $\frac{d}{dx}\sinh(x) = \cosh(x)$

If you see $\sqrt{1+x^2}$, try $x = \sinh(t)$

If you see $\sqrt{x^2-1}$, try $x = \cosh(t)$